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Concentrating solutions of nonlinear fractional Schrödinger equation with potentials

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Abstract

In this paper we study the concentration phenomenon of solutions for the nonlinear fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N,$$

where ε is a positive parameter, $s \in (0, 1)$, $N \geq 2$ and $1 < p < \frac{N+2s}{N-2s}$, $V(x)$ and $K(x)$ are positive smooth functions. Let $\Gamma(x) = [V(x)]^{\frac{p+1}{p-1}-\frac{N}{2s}}[K(x)]^{-\frac{2}{p-1}}$. Under certain assumptions on $V(x)$ and $K(x)$, we show existence and multiplicity of solutions which concentrate near some critical points of $\Gamma(x)$ by a perturbative variational method.

Keywords: Fractional Schrödinger equations; Concentrating solutions; Variational method

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1. Introduction and main results

This paper is devoted to the existence and concentration behavior of semiclassical standing wave solutions for the Schrödinger equations with fractional Laplacian. More precisely, we are concerned with the following equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\varepsilon > 0$ is a positive parameter, $N \geq 2$, $0 < s < 1$ and $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = k_{N,s}\,\text{P.V.} \int\limits_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy. \tag{1.2}$$

The symbol P.V. stands for the Cauchy principal value, and $k_{N,s}$ is a dimensional constant that depends on N and s, precisely given by $k_{N,s} = (\int_{\mathbb{R}^N} \frac{1-\cos\zeta_1}{|\zeta|^{N+2s}} d\zeta)^{-1}$.

The study of elliptic equations involving fractional powers of the Laplacian appears to be important in many physical situations such as combustion, and in dislocations in mechanical systems or in crystals. A basic motivation for the study of Eq. (1.1) arises in the study of standing wave solutions of the type

$$\psi(x,t) = e^{-i\lambda t/\varepsilon} u(x)$$

for the following time-dependent fractional Schrödinger equation

$$i\varepsilon \frac{\partial \psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s \psi + \big(V(x) + \lambda\big)\psi - K(x)|\psi|^{p-1}\psi, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where ε is the Planck constant. Such a ψ solves (1.3) if the standing wave $u(x)$ satisfies (1.1). Eq. (1.3) was introduced by Laskin [19], it is a fundamental equation of fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes. We refer to [9,20] for more physical background.

Very recently, the study on problems of fractional Schrödinger equations has attracted much attention from many mathematicians. Coti Zelati and Nolasco [27] proved the existence of a ground state solution of some fractional Schrödinger equation involving the operator $(-\Delta + d^2)^{\frac{1}{2}}$ with $d > 0$. Cheng [5] obtained the existence of ground state solution of the following equation

$$(-\Delta)^s u + V(x)u - |u|^{p-1}u = 0, \quad x \in \mathbb{R}^N, \tag{1.4}$$

with unbounded potential V. In (1.4), when $V(x) \equiv 1$, Dipierro et al. [14] proved existence and symmetry results for the solutions, and in [17], Felmer et al. studied the same equation with a more general nonlinearity $f(x,u)$, they obtained the existence, regularity and qualitative properties of ground states. Secchi [25] obtained positive solutions of a more general fractional Schrödinger equation by variational method. For other related investigations, one can see [10,26] and references therein.

An important feature of semiclassical states u_ε of (1.1) is that they can concentrate as $\varepsilon \to 0$. We say u_ε concentrates at a point $x_0 \in \mathbb{R}^N$ in the following sense: for any $\sigma > 0$, there exist positive constants ε_0 and ρ such that

$$u_\varepsilon(x) \le \sigma, \quad \text{for } |x-x_0| \ge \varepsilon\rho,\ \forall \varepsilon < \varepsilon_0.$$

As for the classical case $s = 1$, we rewrite (1.1) as follows

$$-\varepsilon^2 \Delta u + V(x)u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N. \tag{1.5}$$

There are many works focusing on Eq. (1.5). When $K(x) \equiv 1$, in the pioneering work [18], Floer and Weinstein studied the case $N = 1$ and $p = 3$, they constructed a positive solution u_ε which concentrates around the critical point of potential $V(x)$, by using the Lyapunov–Schmidt reduction. This method and results have been generalized by [23] to the higher dimensional case. Ambrosetti et al. in [1] proved existence of standing wave solutions by assuming that the potential $V(x)$ has a degenerate local minimum or maximum. In [4], Ambrosetti et al. obtained multiplicity results for some classes of potentials $V(x)$ and $K(x)$ in Eq. (1.5), and proved these solutions concentrations near several non-degenerate critical points of the auxiliary function $A(x) = [V(x)]^{\frac{2p+2+N-Np}{2p-2}}[K(x)]^{-\frac{2}{p-1}}$. Existence of solutions concentrating at one or several points to Eq. (1.5) with potentials vanishing at infinity has been obtained in [2,7,28] and references therein.

In the fractional case $0 < s < 1$, much less is known. When $K(x) = 1$ in (1.1), Chen and Zheng [11] considered the existence and concentration phenomenon under further constraints in the space dimension N and the values of s, by using the Lyapunov–Schmidt reduction method; Dávila et al. [15] proved that if $V(x)$ satisfies

$$V \in C^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \quad \text{and} \quad \inf_{x\in\mathbb{R}^N} V(x) > 0,$$

then (1.1) has multi-peak solutions; in [12], Dávila et al. considered the fractional Schrödinger equation in a bounded domain with zero Dirichlet datum, and built a family of solutions that concentrate at an interior point of the domain.

The aim of this paper is to extend the existence and multiplicity results in [4] for the nonlocal problem (1.1). We obtain the existence and some multiplicity results of (1.1) for all $1 < p < \frac{N+2s}{N-2s}$ under the following assumptions on potentials V and K:

(V1) $V(x) \in C^2(\mathbb{R}^N, \mathbb{R})$, V and its derivatives are uniformly bounded;
(V2) $\inf_{x\in\mathbb{R}^N} V(x) \ge \tau > 0$;
(K1) $K(x) \in C^2(\mathbb{R}^N, \mathbb{R})$, $K(x) > 0$, K and its derivatives are uniformly bounded.

Set

$$\Gamma(x) = \big[V(x)\big]^{\frac{p+1}{p-1}-\frac{N}{2s}}\big[K(x)\big]^{-\frac{2}{p-1}}.$$

By (V1) and (K1), $\Gamma(x)$ is a C^2-smooth function. We say that x_0 is an isolated stable stationary point of $\Gamma(x)$ if the Leray–Schauder index $ind(\nabla\Gamma, x_0, 0) \ne 0$. The index $ind(\nabla\Gamma, x_0, 0) = \lim_{r\to 0} deg(\nabla\Gamma, B_r(x_0), 0)$. It is easy to see that local isolated maxima and minima as well as non-degenerate stationary points are stable.

Now we are ready to state our main results.

Theorem 1.1. *Let* (V1)*,* (V2) *and* (K1) *hold. Suppose that* x_0 *is an isolated stable stationary point of* $\Gamma(x)$*. Then, for* $\varepsilon > 0$ *small, Eq.* (1.1) *has a solution* $u_\varepsilon \in H^s(\mathbb{R}^N)$ *that concentrates at* x_0*.*

We recall that the category $cat(\Sigma, X)$ of a subset Σ of a topological space X is defined as the minimal $k \in \mathbb{N}$ such that Σ is covered by k closed subsets of X which are contractible in X. The *cup long* $l(\Sigma)$ of Σ is defined by

$$l(\Sigma) = 1 + \sup\{k \in \mathbb{N} : \exists \alpha_1, \cdots, \alpha_k \in \breve{H}^*(\Sigma) \setminus 1,\ \alpha_1 \cup \cdots \cup \alpha_k \neq 0\}.$$

If no such class exists, we set $l(\Sigma) = 1$. Where $\breve{H}^*(\Sigma)$ is the Alexander cohomology of Σ with real coefficients and $\cup$ denotes the cup product. In general, one has that $l(\Sigma) \le cat_\Sigma(\Sigma_\delta)$, where Σ_δ denotes its δ neighborhood.

Let Σ be a smooth compact manifold of critical points of $\Gamma(x)$, which is non-degenerate in the sense that for every $x \in \Sigma$ one has that $T_x\Sigma = ker[\Gamma''(x)]$.

Theorem 1.2. *Let* (V1)*,* (V2) *and* (K1) *hold. Suppose that* $\Gamma(x)$ *has a non-degenerate smooth compact manifold of critical points* Σ*. Then, for* $\varepsilon > 0$ *small, Eq.* (1.1) *has at least* $l(\Sigma)$ *solutions that concentrate near points of* Σ*.*

Theorem 1.3. *Let* (V1)*,* (V2) *and* (K1) *hold. Suppose that there is a compact set* Σ *where* $\Gamma(x)$ *achieves an isolated strict local minimum, or maximum. Then, there exists* $\varepsilon_\delta > 0$*, Eq.* (1.1) *has at least* $cat(\Sigma, \Sigma_\delta)$ *solutions that concentrate near points of* Σ *for* $\varepsilon \in (0, \varepsilon_\delta)$*.*

It is worth noting that, a common approach to deal with the fractional nonlocal problem, which was given by Caffarelli and Silvestre [8], is via the Dirichlet–Neumann map transforming (1.1) into a local problem. In this work, we prefer to analyze the problem directly in $H^s(\mathbb{R}^N)$. The proofs of our main results are based on the perturbation method, variational in nature, see [3,4]. The basic idea is to use the non-degeneracy result in [16] to construct solutions of (1.1).

To the best of our knowledge, there is no result on the multiplicity and concentration of solutions for fractional nonlinear Schrödinger equation with potentials. At present paper, we are first devoted to the proof of the existence and concentration of solutions for Eq. (1.1), and then study the multiplicity and concentration of solutions for (1.1). This is the first result for fractional nonlinear Schrödinger equation with potentials. We complement and improve the main results in [11,15], in the sense that we are considering the multiplicity results. Our results are in clear accordance with those for the classical local counterpart, while $s = 1$.

We organize this paper as follows. Section 2 contains some known results. In Section 3, we solve the auxiliary equation which plays a key role in the proofs of our main theorems. In Section 4 the problem is reduced to a finite dimensional variational problem. Finally, in Section 5, we prove our main results.

2. Preliminary results

In this section, we recall some preliminary results which will be useful along the paper. First, we will give some useful facts of the fractional order Sobolev spaces.

For any $0 < s < 1$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\right\},$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy\right)^{\frac{1}{2}},$$

where the term

$$[u]_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy\right)^{\frac{1}{2}}$$

is the so-called Gagliardo semi-norm of u. Let $\mathscr{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$. Indeed, the fractional Laplacian $(-\Delta)^s$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$, as stated in the following

Lemma 2.1. *(See [21].) Let $s \in (0, 1)$ and let $(-\Delta)^s : \mathscr{S} \to L^2(\mathbb{R}^N)$ be the fractional Laplacian operator defined by (1.2). Then, for any $u \in \mathscr{S}$,*

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}\big(|\xi|^{2s}(\mathscr{F}u)\big), \quad \forall \xi \in \mathbb{R}^N.$$

Now, one can see that an alternative definition of the fractional Sobolev space $H^s(\mathbb{R}^N)$ via the Fourier transform is as follows

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s})|\mathscr{F}u|^2 d\xi < +\infty\right\}.$$

It can be proved (Propositions 3.4 and 3.6 of [13]) that

$$2k_{N,s}^{-1} \int_{\mathbb{R}^N} |\xi|^{2s}|\mathscr{F}u|^2 d\xi = 2k_{N,s}^{-1} \big\|(-\Delta)^{\frac{s}{2}} u\big\|^2_{L^2(\mathbb{R}^N)} = [u]^2_{H^s(\mathbb{R}^N)},$$

where $\mathscr{F}$ denotes the Fourier transform. As a consequence, the norms on $H^s(\mathbb{R}^N)$

$$u \mapsto \|u\|_{H^s(\mathbb{R}^N)},$$
$$u \mapsto \Big(\|u\|_{L^2(\mathbb{R}^N)} + \big\|(-\Delta)^{\frac{s}{2}} u\big\|^2_{L^2(\mathbb{R}^N)}\Big)^{\frac{1}{2}},$$
$$u \mapsto \left(\|u\|_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |\xi|^{2s}|\mathscr{F}u|^2 d\xi\right)^{\frac{1}{2}},$$

are all equivalent. For the reader's convenience, we review the main embedding result for this class of fractional Sobolev spaces.

Lemma 2.2. *(See [22,13].) Let $N \geq 1$, $0 < s < 1$ such that $2s < N$. Then there exists a constant $C = C(N, s) > 0$, such that*

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)} \leq C\|u\|_{H^s(\mathbb{R}^N)}$$

for every $u \in H^s(\mathbb{R}^N)$, where $2_s^* = \frac{2N}{N-2s}$ is the fractional critical exponent. Moreover, the embedding $H^s(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is continuous for any $p \in [2, 2_s^*]$, and is locally compact whenever $p \in [2, 2_s^*)$.

We need some tools to handle the nonlocality of the fractional Laplacian. The next lemma below provides a way to manipulate smooth truncations for the fractional Laplacian. First we give the homogeneous Sobolev space

$$H_0^s(\mathbb{R}^N) = \left\{u \in L^{2_s^*}(\mathbb{R}^N) : |\xi|^{\frac{s}{2}}\mathscr{F}(u) \in L^2(\mathbb{R}^N)\right\}.$$

This space can be equivalently defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^2_{H_0^s(\mathbb{R}^N)} = \int\limits_{\mathbb{R}^N} |\xi|^{2s}|\mathscr{F}u|^2 d\xi.$$

Lemma 2.3. *(See [24].) Suppose that $0 < 2s < N$ and $u \in H_0^s(\mathbb{R}^N)$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and for each $r > 0$, $\varphi_r(x) = \varphi(x/r)$. Then*

$$u\varphi_r \to 0 \quad \text{in } H_0^s(\mathbb{R}^N) \text{ as } r \to 0.$$

If, in addition, $\varphi \equiv 1$ in a neighborhood of the origin, then

$$u\varphi_r \to u \quad \text{in } H_0^s(\mathbb{R}^N) \text{ as } r \to \infty.$$

From [16], we know that problem

$$(-\Delta)^s u + u = |u|^{p-1}u, \quad u \in H^s(\mathbb{R}^N), \tag{2.1}$$

has a unique radial positive ground state solution $Q(x)$. The solution $Q(x)$ is smooth and satisfies

$$\frac{\bar{C}}{1 + |x|^{N+2s}} \leq Q(x) \leq \frac{C'}{1 + |x|^{N+2s}}, \quad x \in \mathbb{R}^N \tag{2.2}$$

with some constants $C' \geq \bar{C} > 0$. Moreover, $Q(x)$ is non-degenerate, that is, the kernel of the linearized operator $(-\Delta)^s + 1 - pQ^{p-1}$ is spanned by $\{\frac{\partial Q}{\partial x_i}, i = 1, \cdots, N\}$. Then

$$(-\Delta)^s \frac{\partial Q}{\partial x_i} - pQ^{p-1}\frac{\partial Q}{\partial x_i} = -\frac{\partial Q}{\partial x_i}, \quad i = 1, \cdots, N.$$

We apply Lemma C.2 of [16] to find the following decay estimate

$$|\partial_{x_i} Q| \le \frac{C}{1+|x|^{N+2s}}, \quad i=1,\cdots,N. \tag{2.3}$$

Making the change of variables, we are led to study the following equation

$$(-\Delta)^s u + V(\varepsilon x)u = K(\varepsilon x)|u|^{p-1}u, \quad u \in H^s(\mathbb{R}^N). \tag{2.4}$$

If u_ε is a solution of (2.4) then $u_\varepsilon(\frac{x}{\varepsilon})$ solves (1.1). Solutions of (2.4) are the critical points of

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(\big|(-\Delta)^{\frac{s}{2}}u\big|^2 + V(\varepsilon x)|u|^2\big)dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)|u|^{p+1}dx.$$

Then I_ε is well defined on $H^s(\mathbb{R}^N)$ and belongs to $C^2(H^s(\mathbb{R}^N),\mathbb{R})$ under our assumptions. The solutions of (2.4) will be found near solutions of

$$(-\Delta)^s u + V(\varepsilon \xi)u = K(\varepsilon \xi)|u|^{p-1}u, \quad u \in H^s(\mathbb{R}^N), \tag{2.5}$$

where $\xi \in \mathbb{R}^N$ is regarded as a parameter instead of an independent variable. The functional corresponding to problem (2.5) is

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(\big|(-\Delta)^{\frac{s}{2}}u\big|^2 + V(\varepsilon \xi)|u|^2\big)dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon \xi)|u|^{p+1}dx.$$

Define

$$\alpha(\varepsilon\xi) = \left[\frac{V(\varepsilon\xi)}{K(\varepsilon\xi)}\right]^{\frac{1}{p-1}}, \qquad \beta(\varepsilon\xi) = \big[V(\varepsilon\xi)\big]^{\frac{1}{2s}}, \quad \text{and} \quad U(x) = \alpha(\varepsilon\xi)Q\big(\beta(\varepsilon\xi)x\big).$$

Then, it is easy to check that $U(x)$ satisfies (2.5). Since (2.5) is translation invariant, it follows that any $U_\xi(x) = U(x-\xi)$ is also a solution of (2.5). Let us introduce the critical manifold of J_ε

$$\mathcal{M} = \big\{U_\xi(x) : \xi \in \mathbb{R}^N\big\}.$$

Letting $T_{U_\xi}\mathcal{M}$ denote the tangent space to $\mathcal{M}$ at U_ξ, we know that

$$T_{U_\xi}\mathcal{M} = span\{\partial_{\xi_1}U_\xi, \cdots, \partial_{\xi_N}U_\xi\}.$$

We observe that

$$\partial_{\xi_i}U_\xi(x) = -\partial_{x_i}U_\xi(x) + \varepsilon\partial_{x_i}\alpha(\varepsilon\xi)Q\big[\beta(\varepsilon\xi)(x-\xi)\big] \\ + \varepsilon\alpha(\varepsilon\xi)\partial_{x_i}\beta(\varepsilon\xi)\nabla Q\big(\beta(\varepsilon\xi)(x-\xi)\big)\cdot(x-\xi).$$

By the definition of $\alpha(x)$, $\beta(x)$ and (2.3), we obtain

$$\partial_{\xi_i} U_\xi(x) = -\partial_{x_i} U_\xi(x) + O(\varepsilon). \tag{2.6}$$

3. Solving the auxiliary equation $\pi I'_\varepsilon(U_\xi + \phi) = 0$

In this section, we will find a solution $\phi \in (T_{U_\xi}\mathcal{M})^\perp$ which satisfies the auxiliary equation $\pi I'_\varepsilon(U_\xi + \phi) = 0$, where π denotes the orthogonal projection onto $(T_{U_\xi}\mathcal{M})^\perp$. First of all, we show that the manifold $\mathcal{M}$ is a manifold of approximate critical points for I_ε for $\varepsilon > 0$ small. We denote C and $C_0, C_1, C_2, \cdots$ are positive (possibly different) constants.

Lemma 3.1. *There exists $C_0 > 0$ such that for all $\xi \in \mathbb{R}^N$ and $\varepsilon > 0$ small, we have*

$$\left\| I'_\varepsilon(U_\xi) \right\| \le C_0 \varepsilon. \tag{3.1}$$

Proof. Let us estimate $I'_\varepsilon(U_\xi)[v]$ for any $v \in H^s(\mathbb{R}^N)$. Since $J'_\varepsilon(U_\xi) = 0$, we obtain

$$I'_\varepsilon(U_\xi)[v] = \int_{\mathbb{R}^N} \left[V(\varepsilon x) - V(\varepsilon\xi)\right] U_\xi v\, dx - \int_{\mathbb{R}^N} \left[K(\varepsilon x) - K(\varepsilon\xi)\right] U_\xi^p v\, dx.$$

Using Hölder's inequality, we infer that

$$\begin{aligned} \left| I'_\varepsilon(U_\xi)[v] \right| &\le \|v\|_2 \cdot \left(\int_{\mathbb{R}^N} \left| V(\varepsilon x) - V(\varepsilon\xi) \right|^2 U_\xi^2 dx \right)^{\frac{1}{2}} \\ &\quad + \|v\|_{p+1} \cdot \left(\int_{\mathbb{R}^N} \left| K(\varepsilon x) - K(\varepsilon\xi) \right|^{\frac{p+1}{p}} |U_\xi|^{p+1} dx \right)^{\frac{p}{p+1}}. \end{aligned}$$

By the conditions (V1) and (K1), there exist positive constant V_1 and K_1, and one finds

$$\left| V(\varepsilon x) - V(\varepsilon\xi) \right| \le \varepsilon V_1 |x - \xi|, \qquad \left| K(\varepsilon x) - K(\varepsilon\xi) \right| \le \varepsilon K_1 |x - \xi| \tag{3.2}$$

for any $x, \xi \in \mathbb{R}^N$. It follows from the definition of U_ξ that

$$\int_{\mathbb{R}^N} \left| V(\varepsilon x) - V(\varepsilon\xi) \right|^2 U_\xi^2 dx \le \varepsilon^2 V_1^2 \alpha^2 \beta^{-N-2} \int_{\mathbb{R}^N} |z|^2 Q^2(z) dz,$$

and

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} |K(\varepsilon x) - K(\varepsilon\xi)|^{\frac{p+1}{p}} |U_\xi|^{p+1} dx \right| \\ &\quad \le \varepsilon^{\frac{p+1}{p}} K_1^{\frac{p+1}{p}} \alpha^{p+1} \beta^{-\frac{p+1+Np}{p}} \int_{\mathbb{R}^N} |z|^{\frac{p+1}{p}} Q^{p+1}(z) dz. \end{aligned}$$

By (2.2), we see that

$$\int_{\mathbb{R}^N} |z|^2 Q^2(z)dz \le C, \qquad \int_{\mathbb{R}^N} |z|^{\frac{p+1}{p}} Q^{p+1}(z)dz \le C.$$

Hence, using Lemma 2.2, we obtain (3.1). This completes the proof. □

We now show that $I_\varepsilon''(U_\xi)$ is invertible on $(T_{U_\xi}\mathcal{M})^\perp$, which plays a key role in solving the auxiliary equation.

Lemma 3.2. *There exist $C_1 > 0$ and $T > 0$ such that $\varepsilon > 0$ small and $\xi \in \mathbb{R}^N$ with $|\xi| \le T$. Then $\pi I_\varepsilon''(U_\xi)$ is invertible and $\|[\pi I_\varepsilon''(U_\xi)]^{-1}\| \le C_1$.*

Proof. Let us set $L_{\varepsilon,\xi} : (T_{U_\xi}\mathcal{M})^\perp \to (T_{U_\xi}\mathcal{M})^\perp$:

$$(L_{\varepsilon,\xi}u, v) = \pi I_\varepsilon''(U_\xi)[u, v], \quad \text{for all } u, v \in (T_{U_\xi}\mathcal{M})^\perp.$$

So we only need to show that there exists $k > 0$ such that the interval $(-k, k)$ does not have any eigenvalue of $L_{\varepsilon,\xi}$ on $(T_{U_\xi}\mathcal{M})^\perp$. We decompose $(T_{U_\xi}\mathcal{M})^\perp = X_1 \oplus X_2^\perp$, where X_1 is the space spanned by πU_ξ, $X_2 = U_\xi \oplus T_{U_\xi}\mathcal{M}$.

Since $J'(U_\xi) = 0$, we deduce that

$$\begin{aligned} I_\varepsilon''(U_\xi)[U_\xi, U_\xi] &= \int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{s}{2}} U_\xi\right|^2 + V(\varepsilon x)|U_\xi|^2\right)dx - p\int_{\mathbb{R}^N} K(\varepsilon x)|U_\xi|^{p+1}dx \\ &= \int_{\mathbb{R}^N} \left[V(\varepsilon x) - V(\varepsilon\xi)\right]|U_\xi|^2 dx - p\int_{\mathbb{R}^N} \Bigg[K(\varepsilon x) - K(\varepsilon\xi)|U_\xi|^{p+1}dx \\ &\quad + (1-p)\int_{\mathbb{R}^N} K(\varepsilon\xi)\Bigg]|U_\xi|^{p+1}dx. \end{aligned}$$

So, following the proof of Lemma 3.1, one finds

$$\begin{aligned} I_\varepsilon''(U_\xi)[U_\xi, U_\xi] &\le C\varepsilon + (1-p)\int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{s}{2}} U_\xi\right|^2 + V(\varepsilon\xi)|U_\xi|^2\right)dx \\ &\le C\varepsilon + C(1-p)\|U_\xi\|^2. \end{aligned}$$

Hence, for $\varepsilon > 0$ small, we have

$$I_\varepsilon''(U_\xi)[U_\xi, U_\xi] \le -C\|U_\xi\|^2.$$

Next, we will prove the inequality

$$I_\varepsilon''(U_\xi)[u, u] \ge C_4\|u\|^2, \quad \forall u \in X_2^\perp. \tag{3.3}$$

Define a smooth cut-off function $\eta(x) \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\eta(x) = 1$, $|x| \le 1$; $\eta(x) = 0$, $|x| \ge 2$. For $R > 0$, set $\eta_R(x) = \eta(\frac{x}{R})$. We decompose u as $u = u_1 + u_2$, where $u_1 = \eta_R(x - \xi)u(x)$, $u_2 = (1 - \eta_R(x - \xi))u(x)$.

It is easy to check that

$$I''_\varepsilon(U_\xi)[u,u] = \int\limits_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{s}{2}} u\right|^2 + V(\varepsilon x)|u|^2\right)dx - p\int\limits_{\mathbb{R}^N} K(\varepsilon x)|U_\xi|^{p-1}u^2 dx$$
$$= I''_\varepsilon(U_\xi)[u_1,u_1] + I''_\varepsilon(U_\xi)[u_2,u_2] + 2I''_\varepsilon(U_\xi)[u_1,u_2]. \tag{3.4}$$

Let us estimate $I''_\varepsilon(U_\xi)[u_1,u_2]$. We recall that

$$I''_\varepsilon(U_\xi)[u_1,u_2] = \int\limits_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_1 (-\Delta)^{\frac{s}{2}} u_2 dx + \int\limits_{\mathbb{R}^N} V(\varepsilon x)u_1u_2 dx$$
$$- p\int\limits_{\mathbb{R}^N} K(\varepsilon x)|U_\xi|^{p-1}u_1u_2 dx.$$

Using the boundedness of $K(x)$, Hölder's inequality and Sobolev embeddings, one derives

$$\left|\int\limits_{\mathbb{R}^N} K(\varepsilon x)|U_\xi|^{p-1}u_1u_2 dx\right| \le C\int\limits_{R\le|y|\le 2R} |U(y)|^{p-1}u^2(y+\xi)dy$$
$$\le C\|u\|^2\left(\int\limits_{R\le|y|\le 2R} \left|U(y)\right|^{p+1}dy\right)^{\frac{p-1}{p+1}}.$$

It follows from (2.2) that

$$\left|\int\limits_{\mathbb{R}^N} K(\varepsilon x)|U_\xi|^{p-1}u_1u_2 dx\right| = o_R(1)\|u\|^2.$$

Thus,

$$I''_\varepsilon(U_\xi)[u_1,u_2] \ge C_5\int\limits_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u_1 (-\Delta)^{\frac{s}{2}} u_2 + u_1u_2\right)dx + o_R(1)\|u\|^2. \tag{3.5}$$

Furthermore, in a similar way we infer that

$$I''_\varepsilon(U_\xi)[u_2,u_2] = \int\limits_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{s}{2}} u_2\right|^2 + V(\varepsilon x)|u_2|^2\right)dx - p\int\limits_{\mathbb{R}^N} K(\varepsilon x)|U_\xi|^{p-1}u_2^2 dx$$
$$\ge C_5\|u_2\|^2 + o_R(1)\|u\|^2. \tag{3.6}$$

Finally, we estimate the term $I''_\varepsilon(U_\xi)[u_1, u_1]$. Set $w = u_1 - v$, where v is the projection of u_1 onto X_2. It follows that

$$v = \langle u_1, U_\xi\rangle \frac{U_\xi}{\|U_\xi\|^2} + \sum_{i=1}^{N} \langle u_1, \partial_{\xi_i} U_\xi\rangle \frac{\partial_{\xi_i} U_\xi}{\|\partial_{\xi_i} U_\xi\|^2}. \tag{3.7}$$

It is easy to see that

$$\begin{aligned} I''_\varepsilon(U_\xi)[u_1, u_1] &= \int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{s}{2}} u_1\right|^2 + V(\varepsilon x)|u_1|^2\right)dx - p\int_{\mathbb{R}^N} K(\varepsilon x)|U_\xi|^{p-1} u_1^2 dx \\ &= J''_\varepsilon(U_\xi)[u_1, u_1] + \int_{\mathbb{R}^N} \left[V(\varepsilon x) - V(\varepsilon\xi)\right]|u_1|^2 dx \\ &\quad - p\int_{\mathbb{R}^N} \left[K(\varepsilon x) - K(\varepsilon\xi)\right]|U_\xi|^{p-1}|u_1|^2 dx. \end{aligned} \tag{3.8}$$

By (3.2), the definition of u_1, we can easily get

$$\left|\int_{\mathbb{R}^N} \left[V(\varepsilon x) - V(\varepsilon\xi)\right]|u_1|^2 dx\right| \le \varepsilon V_1 \int_{|y|\le 2R} |y| u^2(y+\xi)dy \le \varepsilon C_6 \|u\|^2,$$

and

$$\begin{aligned} \left|\int_{\mathbb{R}^N} \left[K(\varepsilon x) - K(\varepsilon\xi)\right]|U_\xi|^{p-1}|u_1|^2 dx\right| &\le \varepsilon K_1 \max_{x\in\mathbb{R}^N} U^{p-1} \int_{|y|\le 2R} |y| u^2(y+\xi)dy \\ &\le \varepsilon C_7 \|u\|^2. \end{aligned}$$

From (3.8), we have then that

$$I''_\varepsilon(U_\xi)[u_1, u_1] \ge J''_\varepsilon(U_\xi)[u_1, u_1] - \varepsilon C\|u\|^2. \tag{3.9}$$

On the other hand,

$$J''_\varepsilon(U_\xi)[u_1, u_1] = J''_\varepsilon(U_\xi)[w, w] + J''_\varepsilon(U_\xi)[v, v] + 2J''_\varepsilon(U_\xi)[w, v]. \tag{3.10}$$

Note that the ground state of (2.1) is non-degenerate, and using the same indirect argument as in [11], we deduce that

$$J''_\varepsilon(U_\xi)[w, w] \ge c\|w\|^2. \tag{3.11}$$

Recalling that $u = u_1 + u_2$ and $u \in X_2^{\perp}$, one finds

$$\left|\langle u_1, U_\xi\rangle\right| = \left|\langle u_2, U_\xi\rangle\right| = \left|\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{s}{2}} u_2 (-\Delta)^{\frac{s}{2}} U_\xi + u_2 U_\xi\right) dx\right|.$$

From Lemma 2.3 and the definition of u_2, we infer

$$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_2 (-\Delta)^{\frac{s}{2}} U_\xi dx &= \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} U_\xi dx \\ &\quad - \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_\xi (-\Delta)^{\frac{s}{2}} \left(u\eta_R(x-\xi)\right) dx \\ &= o_R(1).\end{aligned}$$

By Hölder's inequality, Lemma 2.2 and (2.2), we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^N} u_2 U_\xi dx\right| &\le \left|\int_{|y|\ge R} u(y+\xi) U(y) dy\right| \\ &\le C\|u\| \cdot \left(\int_{|y|\ge R} \left|U(y)\right|^2 dy\right)^{\frac{1}{2}} = o_R(1)\|u\|.\end{aligned}$$

Thus,

$$\langle u_1, U_\xi\rangle = o_R(1)\|u\|. \tag{3.12}$$

From (2.3) and (2.6), using the same arguments we get

$$\langle u_1, \partial_{x_i} U_\xi\rangle = o_R(1)\|u\|, \quad i = 1, \cdots, N. \tag{3.13}$$

Putting together (3.7), (3.12) and (3.13), we have

$$\|v\| = o_R(1)\|u\|. \tag{3.14}$$

This implies that

$$\int_{\mathbb{R}^N} K(\varepsilon\xi)|U_\xi|^{p-1}|v|^2 dx = o_R(1)\|u\|^2.$$

It follows from (3.14) and the boundedness of $V(x)$ that

$$J''_\varepsilon(U_\xi)[v, v] = o_R(1)\|u\|^2. \tag{3.15}$$

We can easily obtain

$$J_\varepsilon''(U_\xi)[w,v]=o_R(1)\|u\|^2. \tag{3.16}$$

Putting together (3.11), (3.15), (3.16) and (3.10), we have

$$J_\varepsilon''(U_\xi)[u_1,u_1]\ge c\|w\|^2+o_R(1)\|u\|^2.$$

It follows from the definition of w, (3.14) and (3.9) that

$$I_\varepsilon''(U_\xi)[u_1,u_1]\ge c\|u_1\|^2+o_R(1)\|u\|^2-C\varepsilon\|u\|^2. \tag{3.17}$$

The combination of (3.4)–(3.6) and (3.17) implies

$$I_\varepsilon''(U_\xi)[u,u]\ge C\|u\|^2+o_R(1)\|u\|^2-C\varepsilon\|u\|^2.$$

Letting $\varepsilon\to 0$ and $R\to\infty$, (3.3) follows. This completes the proof. □

Lemma 3.3. *For $\varepsilon>0$ small enough and $|\xi|\le T$, there exists a unique function $\phi_{\varepsilon,\xi}\in(T_{U_\xi}\mathcal{M})^\perp$ such that*

$$\pi I_\varepsilon'(U_\xi+\phi)=0.$$

Moreover, $\phi_{\varepsilon,\xi}$ is of class C^1 with respect to ξ, and for some $C>0$ there holds

$$\|\nabla_\xi\phi_{\varepsilon,\xi}\|\le C\varepsilon^\varrho,\qquad \varrho=\min\{1,p-1\}. \tag{3.18}$$

Proof. For each $\phi\in(T_{U_\xi}\mathcal{M})^\perp$, we apply Taylor's expansion

$$I_\varepsilon'(U_\xi+\phi)=I_\varepsilon'(U_\xi)+I_\varepsilon''(U_\xi)[\phi]+S(U_\xi,\phi),$$

where

$$S(U_\xi,\phi)[v]=-\int_{\mathbb{R}^N}K(\varepsilon x)\big[(U_\xi+\phi)^p-U_\xi^p-pU_\xi^{p-1}\phi\big]v\,dx,$$

for $v\in H^s(\mathbb{R}^N)$. By Lemma 3.2, solving the auxiliary equation $\pi I_\varepsilon'(U_\xi+\phi)=0$ is equivalent to find a fixed point of

$$\phi=F_{\varepsilon,\xi}(\phi),$$

where $F_{\varepsilon,\xi}(\phi)=-L_{\varepsilon,\xi}^{-1}\pi(I_\varepsilon'(U_\xi)+S(U_\xi,\phi))$. For $\gamma=2C_0C_1$, we set

$$B_\varepsilon=\big\{u\in(T_{U_\xi}\mathcal{M})^\perp:\|u\|\le\gamma\varepsilon\big\}.$$

We only need to show that $F_{\varepsilon,\xi}$ is a contraction on B_ε.

Using the Hölder inequality, the Sobolev embeddings and the boundedness of $K(x)$, we observe that

$$\begin{aligned}\big\|S(U_\xi,\phi)[v]\big\| &\le C\int\limits_{\mathbb{R}^N}\big(|\phi|^2+|\phi|^p\big)|v|dx\\ &\le C'\big(\|\phi\|^2+\|\phi\|^p\big)\|v\|. \end{aligned}\tag{3.19}$$

Similarly, we obtain

$$\begin{aligned}&\big\|S(U_\xi,\phi_1)[v]-S(U_\xi,\phi_2)[v]\big\|\\ &\quad\le C''\big(\|\phi_1\|^{p-1}+\|\phi_2\|^{p-1}+\|\phi_1\|+\|\phi_2\|\big)\|\phi_1-\phi_2\|\|v\|. \end{aligned}\tag{3.20}$$

For any $\phi\in B_\varepsilon$, by (3.19), Lemmas 3.1 and 3.2, we conclude

$$\begin{aligned}\big\|F_{\varepsilon,\xi}(\phi)\big\| &\le C_1\big(\big\|I'_\varepsilon(U_\xi)\big\|+\big\|S(U_\xi,\phi)\big\|\big)\\ &\le\big(C_1C_0+C_1C'\gamma^2\varepsilon+C_1C'\gamma^p\varepsilon^{p-1}\big)\varepsilon.\end{aligned}$$

This implies that $F_{\varepsilon,\xi}(\phi)\in B_\varepsilon$ if ε is small enough. From Lemma 3.2 and (3.20), for $\phi_1,\phi_2\in B_\varepsilon$, we find

$$\begin{aligned}\big\|F_{\varepsilon,\xi}(\phi_1)-F_{\varepsilon,\xi}(\phi_2)\big\| &= \big\|L^{-1}_{\varepsilon,\xi}\pi\big(S(U_\xi,\phi_1)-S(U_\xi,\phi_2)\big)\big\|\\ &\le C_1C''\big(\|\phi_1\|^{p-1}+\|\phi_2\|^{p-1}+\|\phi_1\|+\|\phi_2\|\big)\|\phi_1-\phi_2\|\\ &\le 2C_1C''\big(\gamma^{p-1}\varepsilon^{p-1}+\gamma\varepsilon\big)\|\phi_1-\phi_2\|\\ &\le\frac{1}{2}\|\phi_1-\phi_2\|\end{aligned}$$

for $\varepsilon>0$ small enough. Thus $F_{\varepsilon,\xi}$ is a contraction mapping in B_ε, and hence has a unique solution $\phi_{\varepsilon,\xi}\in(T_{U_\xi}\mathcal{M})^\perp$ of $\pi I'_\varepsilon(U_\xi+\phi)=0$ and satisfying

$$\|\phi_{\varepsilon,\xi}\|\le\gamma\varepsilon.\tag{3.21}$$

We prove next that $\xi\mapsto\phi_{\varepsilon,\xi}$ is of class C^1. For fixed $\varepsilon>0$ small, we set

$$H_\varepsilon(\xi,\omega)=\omega-F_{\varepsilon,\xi}(\omega),\qquad \omega\in(T_{U_\xi}\mathcal{M})^\perp\cap B_\varepsilon.$$

Then, $H_\varepsilon(\xi,\phi_{\varepsilon,\xi})=0$. On the other hand,

$$D_\omega H_\varepsilon(\xi,\omega)[\varphi]=\varphi+L^{-1}_{\varepsilon,\xi}\pi\big(\partial_\omega S(U_\xi,\omega)[\varphi]\big)$$

where

$$\partial_\omega S(U_\xi,\omega)[\varphi,v]=-p\int\limits_{\mathbb{R}^N}K(\varepsilon\xi)\big[(U_\xi+\omega)^{p-1}-U_\xi^{p-1}\big]\varphi v\,dx.$$

Similarly, by (3.21), we get

$$\left\|\partial_\omega S(U_\xi,\omega)\right\| \le C\big(\gamma\varepsilon + \gamma^{p-1}\varepsilon^{p-1}\big).$$

Then, $D_\omega H_\varepsilon(\xi,\omega)$ is an invertible operator if ε is sufficiently small; we also note $D_\omega H_\varepsilon(\xi,\omega)$ and $D_\xi H_\varepsilon(\xi,\omega)$ are continuous. Thus, the implicit function theorem implies that the unique fixed point $\phi_{\varepsilon,\xi}$ of $F_{\varepsilon,\xi}$ is of class C^1 with respect to ξ.

Finally, we estimate the gradient $\nabla_\xi \phi_{\varepsilon,\xi}$. We recall that

$$L_{\varepsilon,\xi}\phi_{\varepsilon,\xi} = (-\Delta)^s\phi_{\varepsilon,\xi} + V(\varepsilon x)\phi_{\varepsilon,\xi} - pK(\varepsilon x)U_\xi^{p-1}\phi_{\varepsilon,\xi}.$$

For all $v \in (T_{U_\xi}\mathcal{M})^\perp$, we observe that

$$\begin{aligned}&\int_{\mathbb{R}^N} vL_{\varepsilon,\xi}\phi_{\varepsilon,\xi}\,dx + \int_{\mathbb{R}^N}\big[V(\varepsilon x) - V(\varepsilon\xi)\big]U_\xi v\,dx\\ &\quad + \int_{\mathbb{R}^N}\big[K(\varepsilon\xi) - K(\varepsilon x)\big]U_\xi^p v\,dx - \int_{\mathbb{R}^N} K(\varepsilon x)f(U_\xi,\phi_{\varepsilon,\xi})v\,dx = 0,\end{aligned} \tag{3.22}$$

where

$$f(U_\xi,\phi_{\varepsilon,\xi}) = (U_\xi + \phi_{\varepsilon,\xi})^p - U_\xi^p - pU_\xi^{p-1}\phi_{\varepsilon,\xi}.$$

By differentiation of ξ in (3.22), we have

$$\begin{aligned}&\int_{\mathbb{R}^N} vL_{\varepsilon,\xi}\partial_{\xi_i}\phi_{\varepsilon,\xi}\,dx - p(p-1)\int_{\mathbb{R}^N} K(\varepsilon x)U_\xi^{p-2}\partial_{\xi_i}U_\xi\phi_{\varepsilon,\xi}v\,dx\\ &\quad - \varepsilon\partial_{x_i}V(\varepsilon\xi)\int_{\mathbb{R}^N} U_\xi v\,dx + \int_{\mathbb{R}^N}\big[V(\varepsilon x) - V(\varepsilon\xi)\big]\partial_{\xi_i}U_\xi v\,dx\\ &\quad + p\int_{\mathbb{R}^N}\big[K(\varepsilon\xi) - K(\varepsilon x)\big]U_\xi^{p-1}\partial_{\xi_i}U_\xi v\,dx + \varepsilon\partial_{x_i}K(\varepsilon\xi)\int_{\mathbb{R}^N} U_\xi^p v\,dx\\ &\quad - \int_{\mathbb{R}^N} K(\varepsilon x)(f_{U_\xi}\cdot\partial_{\xi_i}U_\xi + f_{\phi_{\varepsilon,\xi}}\cdot\partial_{\xi_i}\phi_{\varepsilon,\xi})v\,dx = 0,\end{aligned} \tag{3.23}$$

where

$$f_{U_\xi}\cdot\partial_{\xi_i}U_\xi = p(U_\xi + \phi_{\varepsilon,\xi})^{p-1}\partial_{\xi_i}U_\xi - pU_\xi^{p-1}\partial_{\xi_i}U_\xi - p(p-1)U_\xi^{p-2}\partial_{\xi_i}U_\xi\phi_{\varepsilon,\xi}$$

and

$$f_{\phi_{\varepsilon,\xi}}\cdot\partial_{\xi_i}\phi_{\varepsilon,\xi} = p(U_\xi + \phi_{\varepsilon,\xi})^{p-1}\partial_{\xi_i}\phi_{\varepsilon,\xi} - pU_\xi^{p-1}\partial_{\xi_i}\phi_{\varepsilon,\xi}.$$

Denote $L' = L_{\varepsilon,\xi} - G_{\phi_{\varepsilon,\xi}}$, where G_ϕ is defined as $(G_\phi u, v) = \int_{\mathbb{R}^N} K(\varepsilon x) f_\phi u v dx$. We can rewrite (3.23) as follows

$$\begin{aligned}\left(L'\partial_{\xi_i}\phi_{\varepsilon,\xi}, v\right) &= p(p-1)\int_{\mathbb{R}^N} K(\varepsilon x)U_\xi^{p-2}\partial_{\xi_i}U_\xi\phi_{\varepsilon,\xi} v dx + \varepsilon\partial_{x_i}V(\varepsilon\xi)\int_{\mathbb{R}^N} U_\xi v dx \\ &\quad - \int_{\mathbb{R}^N}\left[V(\varepsilon x) - V(\varepsilon\xi)\right]\partial_{\xi_i}U_\xi v dx - p\int_{\mathbb{R}^N}\left[K(\varepsilon\xi) - K(\varepsilon x)\right]U_\xi^{p-1}\partial_{\xi_i}U_\xi v dx \\ &\quad - \varepsilon\partial_{x_i}K(\varepsilon\xi)\int_{\mathbb{R}^N} U_\xi^p v dx + \int_{\mathbb{R}^N} K(\varepsilon x) f_{U_\xi}\partial_{\xi_i}U_\xi v dx.\end{aligned}$$

Using the estimates (2.6) and the arguments already carried out before, one can easily prove that

$$\left|\left(L'\partial_{\xi_i}\phi_{\varepsilon,\xi}, v\right)\right| \le C\left(\varepsilon + \|\phi_{\varepsilon,\xi}\|^\varrho\right)\|v\|.$$

Let us point out that $f_\phi \to 0$ as $\phi \to 0$. Hence, the operator L' is invertible for $\varepsilon > 0$ small by Lemma 3.2. Thus, from (3.21) we obtain

$$\|\partial_{\xi_i}\phi_{\varepsilon,\xi}\| \le C\varepsilon^\varrho.$$

This completes the proof. □

4. The finite-dimensional variational reduction

In this section, we look for critical points of I_ε with the form

$$u = U_\xi + \phi_{\varepsilon,\xi},$$

where $\phi_{\varepsilon,\xi}$ is the function obtained in Lemma 3.3. Define

$$\overline{\mathcal{M}} = \left\{U_\xi + \phi_{\varepsilon,\xi} : \xi \in \mathbb{R}^N\right\}.$$

We show that any constrained critical point u of I_ε on $\overline{\mathcal{M}}$ is a stationary point of I_ε, namely $I_\varepsilon'(u) = 0$. Define the reduced functional $\Phi_\varepsilon(\xi) = I_\varepsilon(U_\xi + \phi_{\varepsilon,\xi})$.

Lemma 4.1. *Assume $\xi_0 \in \mathbb{R}^N$. If $\Phi_\varepsilon'(\xi_0) = 0$, then $I_\varepsilon'(U_{\xi_0} + \phi_{\varepsilon,\xi_0}) = 0$.*

Proof. Let us set $u_\xi = U_\xi + \phi_{\varepsilon,\xi}$. We observe that

$$\begin{aligned}\partial_{\xi_i}\Phi_\varepsilon(\xi) &= \int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_\xi(-\Delta)^{\frac{s}{2}}\partial_{\xi_i}u_\xi dx + \int_{\mathbb{R}^N} V(\varepsilon x)u_\xi\partial_{\xi_i}u_\xi dx \\ &\quad - \int_{\mathbb{R}^N} K(\varepsilon x)|u_\xi|^{p-1}u_\xi\partial_{\xi_i}u_\xi dx \\ &= \left\langle I'(u_\xi), \partial_{\xi_i}u_\xi\right\rangle.\end{aligned}$$

It follows that

$$\big\langle I'(u_\xi), \partial_{\xi_i} u_\xi \big\rangle\big|_{\xi=\xi_0} = 0. \tag{4.1}$$

By (3.18), we have $\partial_{\xi_i}\phi_{\varepsilon,\xi} \to 0$ as $\varepsilon \to 0$, for $i = 1, \cdots, N$. Then, $T_{U_\xi}\mathcal{M}$ is close to $T_{U_\xi}\overline{\mathcal{M}}$ for $\varepsilon > 0$ small enough. By (4.1), we have $I'_\varepsilon(u_{\xi_0})$ is orthogonal to $T_{U_{\xi_0}}\overline{\mathcal{M}}$. On the other hand, Lemma 3.3 implies $I'_\varepsilon(u_{\xi_0}) \in T_{U_{\xi_0}}\mathcal{M}$. Thus, $I'_\varepsilon(U_{\xi_0} + \phi_{\varepsilon,\xi_0}) = 0$. This completes the proof. □

Let us now expand the reduced functional $\Phi_\varepsilon(\xi) = I_\varepsilon(U_\xi + \phi_{\varepsilon,\xi})$. Since U_ξ satisfies Eq. (2.5), we have

$$\begin{aligned}
\Phi_\varepsilon(\xi) &= \frac{1}{2}\int_{\mathbb{R}^N} \big|(-\Delta)^{\frac{s}{2}}(U_\xi + \phi_{\varepsilon,\xi})\big|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|U_\xi + \phi_{\varepsilon,\xi}|^2 dx \\
&\quad - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)|U_\xi + \phi_{\varepsilon,\xi}|^{p+1} dx \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} K(\varepsilon\xi)|U_\xi|^{p+1} dx + \frac{1}{2}\int_{\mathbb{R}^N} \big[V(\varepsilon x) - V(\varepsilon\xi)\big]|U_\xi|^2 dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} \big(\big|(-\Delta)^{\frac{s}{2}}\phi_{\varepsilon,\xi}\big|^2 + V(\varepsilon x)|\phi_{\varepsilon,\xi}|^2\big) dx + I'_\varepsilon(U_\xi)[\phi_{\varepsilon,\xi}] \\
&\quad - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)\big[|U_\xi + \phi_{\varepsilon,\xi}|^{p+1} - U_\xi^{p+1} - (p+1)U_\xi^p \phi_{\varepsilon,\xi}\big] dx \\
&\quad - \frac{1}{p+1}\int_{\mathbb{R}^N} \big[K(\varepsilon x) - K(\varepsilon\xi)\big]|U_\xi|^{p+1} dx.
\end{aligned}$$

It follows from the definition of U_ξ that

$$\int_{\mathbb{R}^N} K(\varepsilon\xi)|U_\xi|^{p+1} dx = \Gamma(\varepsilon\xi)\int_{\mathbb{R}^N} Q^{p+1} dx.$$

We denote $m = (\frac{1}{2} - \frac{1}{p+1})\int_{\mathbb{R}^N} Q^{p+1} dx$,

$$\Upsilon_\varepsilon(\xi) = \frac{1}{2}\int_{\mathbb{R}^N} \big[V(\varepsilon x) - V(\varepsilon\xi)\big]|U_\xi|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} \big[K(\varepsilon x) - K(\varepsilon\xi)\big]|U_\xi|^{p+1} dx,$$

and

$$H_\varepsilon(\xi) = \frac{1}{2}\int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{s}{2}}\phi_{\varepsilon,\xi}\right|^2 + V(\varepsilon x)|\phi_{\varepsilon,\xi}|^2\right)dx$$
$$- \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)\left[|U_\xi + \phi_{\varepsilon,\xi}|^{p+1} - U_\xi^{p+1} - (p+1)U_\xi^p\phi_{\varepsilon,\xi}\right]dx.$$

Then,

$$\Phi_\varepsilon(\xi) = m\Gamma(\varepsilon\xi) + I'_\varepsilon(U_\xi)[\phi_{\varepsilon,\xi}] + \Upsilon_\varepsilon(\xi) + H_\varepsilon(\xi).$$

Lemma 4.2. *The following estimates hold:*

$$\Phi_\varepsilon(\xi) = m\Gamma(\varepsilon\xi) + o(\varepsilon), \tag{4.2}$$

and

$$\nabla_\xi\Phi_\varepsilon(\xi) = \varepsilon m\nabla\Gamma(\varepsilon\xi) + o(\varepsilon). \tag{4.3}$$

Proof. By Lemma 3.1 and (3.21), we get

$$\left|I'_\varepsilon(U_\xi)[\phi_{\varepsilon,\xi}]\right| \le C_0\varepsilon\|\phi_{\varepsilon,\xi}\| \le C_0\gamma\varepsilon^2.$$

Using (3.2) and the polynomial decay of Q, we infer

$$\left|\Upsilon_\varepsilon(\xi)\right| = o(\varepsilon).$$

Using the boundedness of $V(x)$ and (3.21), we obtain

$$\int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{s}{2}}\phi_{\varepsilon,\xi}\right|^2 + V(\varepsilon x)|\phi_{\varepsilon,\xi}|^2\right)dx \le C\|\phi_{\varepsilon,\xi}\|^2 \le C\gamma^2\varepsilon^2.$$

Furthermore, by the boundedness of $K(x)$, the Sobolev embeddings and (3.21), we get

$$\left|\int_{\mathbb{R}^N} K(\varepsilon x)\left[|U_\xi + \phi_{\varepsilon,\xi}|^{p+1} - U_\xi^{p+1} - (p+1)U_\xi^p\phi_{\varepsilon,\xi}\right]dx\right| \le C\left(\gamma^2\varepsilon^2 + \gamma^{p+1}\varepsilon^{p+1}\right).$$

Hence, we observe that

$$\left|H_\varepsilon(\xi)\right| = o(\varepsilon).$$

Then, (4.2) holds.

Next, we estimate the derivatives of Φ_ε with respect to ξ. We compute

$$\partial_{\xi_i}\Phi_\varepsilon(\xi) = m\varepsilon\partial_{x_i}\Gamma(\varepsilon\xi) + \partial_{\xi_i}\left(I'_\varepsilon(U_\xi)[\phi_{\varepsilon,\xi}]\right) + \partial_{\xi_i}\Upsilon_\varepsilon(\xi) + \partial_{\xi_i}H_\varepsilon(\xi). \tag{4.4}$$

By a direct calculation, we find

$$\partial_{\xi_i}\left(I'_\varepsilon(U_\xi)[\phi_{\varepsilon,\xi}]\right) = I'_\varepsilon(U_\xi)[\partial_{\xi_i}\phi_{\varepsilon,\xi}] + I''_\varepsilon(U_\xi)[\partial_{\xi_i}U_\xi, \phi_{\varepsilon,\xi}]. \tag{4.5}$$

From Lemma 3.1 and (3.18), we obtain

$$\left\| I'_\varepsilon(U_\xi)[\partial_{\xi_i}\phi_{\varepsilon,\xi}]\right\| \le C\varepsilon^{\varrho+1}. \tag{4.6}$$

Since U_ξ satisfies (2.5), by the non-degeneracy of Q, we have now that

$$\begin{aligned}
I''_\varepsilon(U_\xi)[\partial_{\xi_i}U_\xi, \phi_{\varepsilon,\xi}] &= I''_\varepsilon(U_\xi)[\partial_{\xi_i}U_\xi, \phi_{\varepsilon,\xi}] + J''_\varepsilon(U_\xi)[\partial_{x_i}U_\xi, \phi_{\varepsilon,\xi}] \\
&= \int_{\mathbb{R}^N} \phi_{\varepsilon,\xi}(-\Delta)^s(\partial_{\xi_i}U_\xi + \partial_{x_i}U_\xi)dx \\
&\quad + \int_{\mathbb{R}^N} \left[V(\varepsilon x)\partial_{\xi_i}U_\xi + V(\varepsilon\xi)\partial_{x_i}U_\xi\right]\phi_{\varepsilon,\xi}dx \\
&\quad - p\int_{\mathbb{R}^N} \left[K(\varepsilon x)\partial_{\xi_i}U_\xi + K(\varepsilon\xi)\partial_{x_i}U_\xi\right]U_\xi^{p-1}\phi_{\varepsilon,\xi}dx.
\end{aligned}$$

Using (3.21) and (2.6), we find

$$\begin{aligned}
\int_{\mathbb{R}^N} \phi_{\varepsilon,\xi}(-\Delta)^s(\partial_{\xi_i}U_\xi + \partial_{x_i}U_\xi)dx &= \int_{\mathbb{R}^N} (\partial_{\xi_i}U_\xi + \partial_{x_i}U_\xi)(-\Delta)^s\phi_{\varepsilon,\xi}dx \\
&= O(\varepsilon)\int_{\mathbb{R}^N} (-\Delta)^s\phi_{\varepsilon,\xi}dx \\
&\le O(\varepsilon)\|\phi_{\varepsilon,\xi}\|^2 \le C\varepsilon^3.
\end{aligned}$$

Moreover, by (3.21), (2.6) and arguing as in the proof of Lemma 3.1, one easily obtains

$$\left|\int_{\mathbb{R}^N} \left[V(\varepsilon x)\partial_{\xi_i}U_\xi + V(\varepsilon\xi)\partial_{x_i}U_\xi\right]\phi_{\varepsilon,\xi}dx\right| = o(\varepsilon),$$

$$\left|\int_{\mathbb{R}^N} \left[K(\varepsilon x)\partial_{\xi_i}U_\xi + K(\varepsilon\xi)\partial_{x_i}U_\xi\right]U_\xi^{p-1}\phi_{\varepsilon,\xi}dx\right| = o(\varepsilon).$$

Hence, we have

$$I''_\varepsilon(U_\xi)[\partial_{\xi_i}U_\xi, \phi_{\varepsilon,\xi}] = o(\varepsilon).$$

It follows from (4.5) and (4.6) that

$$\left|\partial_{\xi_i}\left(I'_\varepsilon(U_\xi)[\phi_{\varepsilon,\xi}]\right)\right| = o(\varepsilon). \tag{4.7}$$

Let us compute

$$\partial_{\xi_i}\Upsilon_\varepsilon(\xi)=\int_{\mathbb{R}^N}\big[V(\varepsilon x)-V(\varepsilon\xi)\big]U_\xi\partial_{\xi_i}U_\xi dx-\frac{\varepsilon}{2}\partial_{x_i}V(\varepsilon\xi)\int_{\mathbb{R}^N}U_\xi^2dx$$
$$+\frac{\varepsilon}{p+1}\partial_{x_i}K(\varepsilon\xi)\int_{\mathbb{R}^N}U_\xi^{p+1}dx-\int_{\mathbb{R}^N}\big[K(\varepsilon x)-K(\varepsilon\xi)\big]U_\xi^p\partial_{\xi_i}U_\xi dx.$$

By the boundedness of ∇V and ∇K, we have

$$\left|\frac{\varepsilon}{2}\partial_{x_i}V(\varepsilon\xi)\int_{\mathbb{R}^N}U_\xi^2dx\right|=o(\varepsilon),\qquad\left|\frac{\varepsilon}{p+1}\partial_{x_i}K(\varepsilon\xi)\int_{\mathbb{R}^N}U_\xi^{p+1}dx\right|=o(\varepsilon).$$

It follows from the similar proof of Lemma 3.1 that

$$\big|\partial_{\xi_i}\Upsilon_\varepsilon(\xi)\big|=o(\varepsilon).\tag{4.8}$$

We observe that

$$\partial_{\xi_i}H_\varepsilon(\xi)=\int_{\mathbb{R}^N}\big((-\Delta)^{\frac{s}{2}}\phi_{\varepsilon,\xi}(-\Delta)^{\frac{s}{2}}\partial_{\xi_i}\phi_{\varepsilon,\xi}+V(\varepsilon x)\phi_{\varepsilon,\xi}\partial_{\xi_i}\phi_{\varepsilon,\xi}\big)dx$$
$$-\int_{\mathbb{R}^N}K(\varepsilon x)\big[|U_\xi+\phi_{\varepsilon,\xi}|^{p-1}(U_\xi+\phi_{\varepsilon,\xi})(\partial_{\xi_i}U_\xi+\partial_{\xi_i}\phi_{\varepsilon,\xi})$$
$$-U_\xi^p\partial_{\xi_i}U_\xi-pU_\xi^{p-1}\partial_{\xi_i}U_\xi\phi_{\varepsilon,\xi}-U_\xi^p\partial_{\xi_i}\phi_{\varepsilon,\xi}\big]dx.$$

From the boundedness of $V(x)$, (3.18) and (3.21), we deduce that

$$\left|\int_{\mathbb{R}^N}\big((-\Delta)^{\frac{s}{2}}\phi_{\varepsilon,\xi}(-\Delta)^{\frac{s}{2}}\partial_{\xi_i}\phi_{\varepsilon,\xi}+V(\varepsilon x)\phi_{\varepsilon,\xi}\partial_{\xi_i}\phi_{\varepsilon,\xi}\big)dx\right|$$
$$\le C\big(\|\phi_{\varepsilon,\xi}\|^2+\|\partial_{\xi_i}\phi_{\varepsilon,\xi}\|^2\big)\le C\big(\varepsilon^2+\varepsilon^{2\varrho}\big).$$

Using the Hölder inequality, Sobolev embeddings, (3.21) and (2.6), we infer

$$\left|\int_{\mathbb{R}^N}K(\varepsilon x)\big[|U_\xi+\phi_{\varepsilon,\xi}|^{p-1}(U_\xi+\phi_{\varepsilon,\xi})-U_\xi^p-pU_\xi^{p-1}\phi_{\varepsilon,\xi}\big]\partial_{\xi_i}U_\xi dx\right|$$
$$\le C\big(\|\phi_{\varepsilon,\xi}\|^2+\|\phi_{\varepsilon,\xi}\|^p\big)\le C\big(\varepsilon^2+\varepsilon^p\big),$$

and

$$\left|\int_{\mathbb{R}^N}K(\varepsilon x)\big[|U_\xi+\phi_{\varepsilon,\xi}|^{p-1}(U_\xi+\phi_{\varepsilon,\xi})-U_\xi^p\big]\partial_{\xi_i}\phi_{\varepsilon,\xi}dx\right|\le C\big(\varepsilon^{1+\varrho}+\varepsilon^{p+\varrho}\big).$$

Thus,

$$\left|\partial_{\xi_i} H_\varepsilon(\xi)\right| = o(\varepsilon). \tag{4.9}$$

Putting together (4.7)–(4.9) with (4.4), yields the estimate (4.3). This concludes the proof. □

5. The proof of main results

In this section, we will show our main results of the present paper. First, we give the proof of the existence of solution of (1.1)

Proof of Theorem 1.1. Set $\widetilde{\Phi}_\varepsilon(\xi) = \Phi_\varepsilon(\frac{\xi}{\varepsilon})$. It follows from (4.2) that

$$\widetilde{\Phi}_\varepsilon(\xi) = m\Gamma(\xi) + o(\varepsilon).$$

Let $x_0 \in \mathbb{R}^N$ be an isolated minimum of $\Gamma(x)$, that is, for some $\delta > 0$,

$$\Gamma(x) > \Gamma(x_0) \quad \text{for all } 0 < |x - x_0| < \delta. \tag{5.1}$$

It is easy to see that

$$\widetilde{\Phi}_\varepsilon(\xi) - \widetilde{\Phi}_\varepsilon(x_0) = m\big(\Gamma(\xi) - \Gamma(x_0)\big) + o(\varepsilon).$$

By (5.1), we observe that $\Gamma(\xi) - \Gamma(x_0) \geq \sigma > 0$ for all $0 < |\xi - x_0| = \rho \leq \delta$. Hence,

$$\inf_{|\xi - x_0| = \rho} \widetilde{\Phi}_\varepsilon(\xi) > \widetilde{\Phi}_\varepsilon(x_0)$$

for all $0 < |\xi - x_0| = \rho \leq \delta$ and $\varepsilon > 0$ small. It follows that $\widetilde{\Phi}_\varepsilon(\xi)$ has a critical point ξ_ε satisfying $\xi_\varepsilon \to x_0$ as $\varepsilon \to 0$. Then, $\frac{\xi_\varepsilon}{\varepsilon}$ is a critical point of Φ_ε. From Lemma 4.1 we see that $U(x - \frac{\xi_\varepsilon}{\varepsilon}) + \phi_{\varepsilon,\xi_\varepsilon}$ is a critical point of I_ε and hence a solution of (2.4). Thus, (1.1) has a solution u_ε of the form

$$u_\varepsilon(x) = U\left(\frac{x - \xi_\varepsilon}{\varepsilon}\right) + \phi_{\varepsilon,\xi_\varepsilon}$$

with $\xi_\varepsilon \to x_0$ and $\|\phi_{\varepsilon,\xi_\varepsilon}\| \to 0$ as $\varepsilon \to 0$. The case of a maximum requires merely obvious changes.

Now, we consider $x_0 \in \mathbb{R}^N$ which is an isolated stable critical point of $\Gamma(x)$. By (4.3), we infer that for $\varepsilon > 0$ small,

$$ind(\nabla_\xi \widetilde{\Phi}_\varepsilon, x_0, 0) = ind(\nabla_\xi \Gamma, x_0, 0) \neq 0.$$

This implies that $\widetilde{\Phi}_\varepsilon(\xi)$ has a critical point ξ_ε satisfying $\xi_\varepsilon \to x_0$ as $\varepsilon \to 0$. Similarly, we deduce that (1.1) has a solution u_ε which concentrates at x_0 as $\varepsilon \to 0$. □

In order to prove the problem (1.1) has $l(\Sigma)$ solutions, we recall a result of [6].

Theorem 5.1. *Let $f \in C^2(\mathbb{R}^N, \mathbb{R})$ and suppose that $\Sigma \subset \mathbb{R}^N$ is a non-degenerate compact manifold of critical points of f. Let $\mathcal{N}$ be a neighborhood of Σ and let $g \in C^1(\mathbb{R}^N, \mathbb{R})$. If $\|f - g\|_{C^1}$ is sufficiently small, the function g has at least $l(\Sigma)$ critical points in $\mathcal{N}$.*

Proof of Theorem 1.2. We let $f(\xi) = m\Gamma(\xi)$, Σ is obviously a non-degenerate smooth compact manifold of f. Fix a neighborhood $\mathcal{N}$ of Σ, and set $g(\xi) = \widetilde{\Phi_\varepsilon}(\xi)$. By Lemma 4.2, we know that $\|f - g\|_{C^1} \to 0$ as $\varepsilon \to 0$. From Theorem 5.1, we can deduce the existence of at least $l(\Sigma)$ critical points of g in $\mathcal{N}$ when $\varepsilon \to 0$ is small enough. Now let $\xi_{\varepsilon,i} \in \mathcal{N}$ be any of those critical points. Then, $\frac{\xi_{\varepsilon,i}}{\varepsilon}$ is a critical point of Φ_ε. It follows from Lemma 4.1 that $U(x - \frac{\xi_\varepsilon}{\varepsilon}) + \phi_{\varepsilon,\xi_\varepsilon}$ is a critical point of I_ε and hence a solution of (2.4). Thus, (1.1) has a solution $u_{\varepsilon,i}$ of the form

$$u_{\varepsilon,i}(x) = U\left(\frac{x - \xi_{\varepsilon,i}}{\varepsilon}\right) + \phi_{\varepsilon,\xi_{\varepsilon,i}}$$

with $\|\phi_{\varepsilon,\xi_{\varepsilon,i}}\| \to 0$ and any $\xi_{\varepsilon,i} \to \widetilde{\xi_i} \in \mathcal{N}$ as $\varepsilon \to 0$. Using (4.3), we infer that $\widetilde{\xi_i}$ is a stationary point of Γ. Then, taking $\mathcal{N}$ possibly smaller, it follows that $\widetilde{\xi_i} \in \Sigma$, and hence $u_{\varepsilon,i}(x)$ concentrates near a point of Σ. □

Proof of Theorem 1.3. We consider the case that $\Gamma(x)$ has an isolated strict local minimum in compact set Σ, that is, there exists $\delta > 0$ satisfying

$$\theta = \inf\{\Gamma(x) : x \in \partial\Sigma_\delta\} > \mu = \Gamma(x)|_\Sigma, \tag{5.2}$$

where Σ_δ is the δ-neighborhood of Σ. Set $X = \{\xi \in \Sigma_\delta : \widetilde{\Phi_\varepsilon}(\xi) \le \frac{m(\theta+\mu)}{2}\}$. By (5.2), there exists $\varepsilon_\delta > 0$ such that for all $\varepsilon \in (0, \varepsilon_\delta)$, we have

$$\Sigma \subset X \subset \Sigma_\delta. \tag{5.3}$$

Let $\{\xi_n\} \subset X$, suppose that there exists $\xi_0 \in \mathbb{R}^N$ such that $\xi_n \to \xi_0$ as $n \to \infty$. Then, $\widetilde{\Phi_\varepsilon}(\xi_0) \le \frac{m(\theta+\mu)}{2}$ and $\xi_0 \in \overline{\Sigma_\delta}$. If $\xi_0 \in \partial\Sigma_\delta$, using (5.2) we observe that

$$\widetilde{\Phi_\varepsilon}(\xi_0) \ge m\theta + o(\varepsilon) > \frac{m(\theta + \mu)}{2},$$

a contradiction. We conclude $\xi_0 \in \Sigma$, and hence X is compact. Then Lusternik–Schnirelman theorems imply that $\widetilde{\Phi_\varepsilon}(\xi)$ has at least $cat(X, \Sigma_\delta)$ critical points on Σ_δ. It follows from (5.3) and the properties of the category that $cat(X, \Sigma_\delta) \ge cat(\Sigma, \Sigma_\delta)$. Thus, we obtain $\widetilde{\Phi_\varepsilon}(\xi)$ has at least $cat(\Sigma, \Sigma_\delta)$ critical points. Using arguments already carried out in the proof of Theorem 1.2, we conclude that Eq. (1.1) has at least $cat(\Sigma, \Sigma_\delta)$ solutions that concentrate near points of Σ for $\varepsilon \in (0, \varepsilon_\delta)$. Similarly, we deduce the case of a maximum. □

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