Higher order strong approximations of semilinear stochastic wave equation with additive space-time white noise

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Consider semilinear stochastic wave equations (SWEs) driven by additive space-time white noise,

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x, u) + \dot{W}, \quad t \in (0, T], \ x \in (0, 1), \\
u(0, x) = u_0(x), \ \frac{\partial u}{\partial t}(0, x) = v_0(x), \ x \in (0, 1), \\
u(t, 0) = u(t, 1) = 0, \ t > 0,
\end{cases}
\]  

(1.1)

where \( T \in (0, \infty) \) and \( f: [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a smooth nonlinear function satisfying
\begin{align}
|f(x, z)| & \leq L(|z| + 1), \quad \text{(1.2)} \\
|\frac{\partial f}{\partial z}(x, z)| & \leq L, \quad |\frac{\partial^2 f}{\partial x \partial z}(x, z)| \leq L, \quad \text{and} \quad |\frac{\partial^2 f}{\partial z^2}(x, z)| \leq L \quad \text{(1.3)}
\end{align}

for all $x \in [0, 1]$, $z \in \mathbb{R}$ and some constant $L > 0$. The initial data $u_0$ and $v_0$ are random variables that will be specified later. The forcing term $\dot{W}$ is a space-time white noise (see below), which best models the fluctuations generated by microscopic effects in a homogeneous physical system.
Convergence results of numerical methods for SWEs in literature


Spatial semi-discretizations of SWE subject to multiplicative space-time white noise finite difference method

strong convergence rate of order: $\frac{1}{3}$

SWE with additive noise

spectral Galerkin method in spatial approximation

strong convergence rate of order is improved to $\frac{1}{2} - \epsilon$ for arbitrarily small $\epsilon > 0$. 

fully discrete finite difference scheme, based on leapfrog discretization

strong convergence order in both time and space: $\frac{1}{2}$. 
To the best of our knowledge, we have not found any numerical method that strongly converges with a rate faster than $\frac{1}{2}$ in the literature. This seems to be an order barrier. In fact, the limit on the convergence rate of numerical schemes for SWEs driven by space-time white noise, has been established by Walsh in the sense that no scheme based on the basic increments of white noise strongly converges at a rate faster than $\frac{1}{2}$. An interesting question thus arises as to whether it is possible to overcome the order barrier.
In this work, we provide a positive answer to this question by designing two fully discrete schemes for the SWE (1.1), which enjoy a strong convergence order greater than $\frac{1}{2}$. More precisely, we spatially discretize (1.1) by a spectral Galerkin method, and then propose two exponential time integrators involving two linear functionals of the noise.
As shown in the main convergence result (Theorem 5), under the conditions (1.2) and (1.3) the proposed fully discrete schemes strongly converge with order $\frac{1}{2} - \epsilon$ in space and order $1 - \epsilon$ in time for arbitrarily small $\epsilon > 0$. Compared with existing schemes mentioned earlier, the proposed schemes are easy to implement and produce significant improvement on the computational efficiency.
Let \((U, \langle \cdot, \cdot \rangle, \| \cdot \|)\) and \((H, (\cdot, \cdot), \|\| \cdot \||\|)\) be two separable Hilbert spaces. By \(\mathcal{L}(U, H)\) we denote the Banach space of bounded linear operators from \(U\) to \(H\) and for short we write \(\mathcal{L}(U) := \mathcal{L}(U, U)\). Additionally, we need the Banach space of Hilbert-Schmidt operators, denoted by \(\mathcal{L}_2(U, H)\), equipped with the norm

\[
\|\Gamma\|_{\mathcal{L}_2(U, H)} = \left( \sum_{i=1}^{\infty} \|\|\Gamma \eta_i\||^2 \right)^{1/2},
\]  

(2.1)

where \(\{\eta_i\}_{i \in \mathbb{N}}\) is an orthonormal basis of \(U\) and the norm does
not depend on the particular choice of the basis. For brevity, we write $\mathcal{L}_2(U) := \mathcal{L}_2(U, U)$. If $\Gamma_1 \in \mathcal{L}(U, H)$ and $\Gamma_2 \in \mathcal{L}_2(U)$, then $\Gamma_1 \Gamma_2 \in \mathcal{L}_2(U, H)$, and

$$\|\Gamma_1 \Gamma_2\|_{\mathcal{L}_2(U, H)} \leq \|\Gamma_1\|_{\mathcal{L}(U, H)} \cdot \|\Gamma_2\|_{\mathcal{L}_2(U)}. \quad (2.2)$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ and by $L^p(\Omega, U)$ we denote the space of $U$-valued integrable random variables with the norm defined by

$$\|\varphi\|_{L^p(\Omega, U)} = \left(\mathbb{E}\left[\|\varphi\|^p\right]\right)^{\frac{1}{p}} < \infty \text{ for any } p \geq 2.$$
We take $U := L^2((0, 1), \mathbb{R})$ to denote the space of real-valued square integrable functions, equipped with the usual norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$. Let $-\Lambda = \Delta : \mathcal{D}(\Lambda) \subset U \to U$ be the Laplace operator with $\mathcal{D}(\Lambda) = H^2(0, 1) \cap H^1_0(0, 1)$, where $H^m(0, 1)$ denote the standard Sobolev spaces of integer order $m \geq 1$ and $H^1_0(0, 1) := \{ \varphi \in H^1(0, 1) : \varphi(0) = \varphi(1) = 0 \}$. Then $\Lambda$ is a densely defined, self-adjoint, positive operator with compact inverse.
Moreover, the eigenvalue problem

$$\Lambda e_i = \lambda_i e_i, \; i \in \mathbb{N}$$ \hspace{1cm} (2.3)

provides an orthonormal basis \(\{e_i = \sqrt{2} \sin(i\pi x), \; x \in (0, 1)\}\) for \(U\) and an increasing sequence of eigenvalues \(\lambda_i = \pi^2 i^2, \; i \in \mathbb{N}\).

Additionally, let \(F : U \to U\) be a Nemytskii operator associated to \(f\) as in (1.1), defined by

$$F(\varphi)(x) = f(x, \varphi(x)), \; x \in (0, 1), \; \varphi \in U.$$ \hspace{1cm} (2.4)
Now one can rewrite (1.1) as an abstract form in Itô’s sense

\[
\begin{align*}
    \frac{du}{dt} &= -\lambda u dt + F(u) dt + dW(t), \quad t \in (0, T], \\
    u(0) &= u_0, \quad \dot{u}(0) = v_0,
\end{align*}
\]  

(2.5)

where $u$ is regarded as a $U$-valued stochastic process and $\dot{u}$ stands for the time derivative of $u$. 

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The driven stochastic process $W(t)$ is a cylindrical $I$-Wiener process with respect to $\{\mathcal{F}_t\}_{0 \leq t \leq T}$, which can be represented as follows:

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t)e_i, \quad t \in [0, T], \quad (2.6)$$

where $\{\beta_i(t)\}_{i \in \mathbb{N}}$ are independent real-valued Brownian motions and $\{e_i\}_{i \in \mathbb{N}}$ are the eigenvectors of $\Lambda$ defined by (2.3).
Moreover, note that the derivative operators of $F$ are given by

\[ F'(\varphi)(\psi)(x) = \frac{\partial f}{\partial z}(x, \varphi(x)) \cdot \psi(x), \quad x \in (0, 1), \quad (2.7) \]

\[ F''(\varphi)(\psi_1, \psi_2)(x) = \frac{\partial^2 f}{\partial z^2}(x, \varphi(x)) \cdot \psi_1(x) \cdot \psi_2(x), \quad x \in (0, 1), \quad (2.8) \]

for all $\varphi, \psi, \psi_1, \psi_2 \in U$. It is worthwhile to keep in mind that the derivative operators $F'(\varphi), F''(\varphi), \varphi \in U$ defined in the above way are self-adjoint.
The Nemytskii operator $F$ satisfies

$$\|F(\varphi)\| \leq \sqrt{2L}(\|\varphi\| + 1), \quad (2.9)$$

$$\|F(\varphi_1) - F(\varphi_2)\| \leq L\|\varphi_1 - \varphi_2\| \quad (2.10)$$

for all $\varphi, \varphi_1, \varphi_2 \in U$. Also, it is obvious that

$$\left\| \Lambda^{\frac{\beta-1}{2}} \right\|_{L^2(U)}^2 = \sum_{i=1}^{\infty} \left\| \Lambda^{\frac{\beta-1}{2}} e_i \right\|^2 = \sum_{i=1}^{\infty} \frac{\pi^2(\beta-1)}{i^2(1-\beta)} \leq \overline{c}_\beta < \infty, \quad (2.11)$$

for any $\beta < \frac{1}{2}$. 
In order to define the mild solution of (2.5) appropriately, we shall reformulate (2.5) as a stochastic evolution equation in a new Hilbert space $H$ to fall into the semigroup framework. To this end, we need additional spaces and notations. The above setting enables us to define fractional powers of $\Lambda$ in a simple way. Accordingly, we introduce the separable Hilbert space

$$\dot{H}^\alpha := D(\Lambda^{\frac{\alpha}{2}}), \alpha \in \mathbb{R},$$

equipped with the inner product

$$\langle \varphi, \psi \rangle_\alpha := \langle \Lambda^{\frac{\alpha}{2}} \varphi, \Lambda^{\frac{\alpha}{2}} \psi \rangle = \sum_{i=1}^{\infty} \lambda_i^{\alpha} \langle \varphi, e_i \rangle \langle \psi, e_i \rangle, \quad \varphi, \psi \in \dot{H}^\alpha,$$

(2.12)
where \( \{(\lambda_i, e_i)\}_{i=1}^{\infty} \) are the eigenpairs of \( \Lambda \). The corresponding norm is defined by \( \|\varphi\|_\alpha = \sqrt{\langle \varphi, \varphi \rangle_\alpha} \) for \( \varphi \in \dot{H}_\alpha \). Then \( \dot{H}^0 = U \) and \( \dot{H}^\alpha \subset \dot{H}^\beta \) if \( \alpha \geq \beta \). Moreover, \( \dot{H}^{-\gamma} \) can be identified with the dual space \( (\dot{H}^\gamma)^* \) for \( \gamma > 0 \).

Further, we introduce the product space \( H^\alpha := \dot{H}^\alpha \times \dot{H}^{\alpha-1} \), \( \alpha \in \mathbb{R} \), endowed with the inner product

\[
(Y, \hat{Y})_\alpha := \langle \varphi, \hat{\varphi} \rangle_\alpha + \langle \psi, \hat{\psi} \rangle_{\alpha-1}, \quad Y = (\varphi, \psi)^T, \quad \hat{Y} = (\hat{\varphi}, \hat{\psi})^T,
\]

\[ (2.13) \]
and the usual norm

\[ \|\|Y\|\|^2_\alpha := \|\varphi\|^2_\alpha + \|\psi\|^2_{\alpha-1}, \quad Y = (\varphi, \psi)^T. \quad (2.14) \]

It is easy to check that \((H^\alpha, (\cdot, \cdot)_\alpha), \alpha \in \mathbb{R}\), is a separable Hilbert space. For the special case \(\alpha = 0\), we denote

\[ H := H^0 = \dot{H}^0 \times \dot{H}^{-1}, \quad (\cdot, \cdot) := (\cdot, \cdot)_0, \quad \text{and} \quad \|\| \cdot \|\|_0 := \|\| \cdot \|\|_0. \]
At this point, we introduce the velocity of the solution $u$, denoted by $v = \dot{u}$, and formally transform (2.5) into the following Cauchy problem

$$
\begin{cases}
    \mathrm{d}X(t) = AX(t)\mathrm{d}t + F(X)\mathrm{d}t + B\mathrm{d}W(t), & t \in (0, T], \\
    X(0) = X_0,
\end{cases}
$$

(2.15)

where $X_0 = (u_0, v_0)^T$ and

$$
X = \begin{bmatrix} u \\ v \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix}, \quad F(X) = \begin{bmatrix} 0 \\ F(u) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}.
$$

(2.16)
Here $X_0$ is assumed to be an $\mathcal{F}_0$-measurable $H$-valued random variable and $B$ is considered as an operator from $\dot{H}^{-1}$ to $H$.

From now on, we regard $\Lambda$ as an operator from $\dot{H}^1$ to $\dot{H}^{-1}$, defined by $(\Lambda \varphi)(\psi) = \langle \nabla \varphi, \nabla \psi \rangle$ for $\varphi, \psi \in \dot{H}^1$ and define the domain of $A$ by

$$
\mathcal{D}(A) = \left\{ Y = (\varphi, \psi)^T \in H : AY = \begin{bmatrix} \psi \\ -\Lambda \varphi \end{bmatrix} \in \dot{H} = \dot{H}^0 \times \dot{H}^{-1} \right\} = H^1 = \dot{H}^1 \times \dot{H}^0.
$$
Then the operator $A$ is the generator of a strongly continuous semigroup $E(t), t \geq 0$ on $H$, that can be written as

$$E(t) = e^{tA} = \begin{bmatrix} C(t) & \Lambda^{-\frac{1}{2}} S(t) \\ -\Lambda^{\frac{1}{2}} S(t) & C(t) \end{bmatrix}. \quad (2.17)$$
Here $C(t) = \cos(t\Lambda^{1/2})$ and $S(t) = \sin(t\Lambda^{1/2})$ are the so-called cosine and sine operators, which can be expressed in terms of the eigenpairs $\{\lambda_i, e_i\}_{i \in \mathbb{N}}$:

$$
C(t)\varphi = \sum_{i=1}^{\infty} \cos(t\lambda_i^{1/2}) \langle \varphi, e_i \rangle e_i, \quad S(t)\varphi = \sum_{i=1}^{\infty} \sin(t\lambda_i^{1/2}) \langle \varphi, e_i \rangle e_i
$$

for $t \geq 0, \varphi \in \dot{H}^{-1}$. 

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Theorem 1

Suppose conditions (1.2) and (1.3) are fulfilled, let $W(t), t \in [0, T]$ be the cylindrical $I$-Wiener process represented by (2.6), and let $X_0$ be an $\mathcal{F}_0$-measurable $H$-valued random variable satisfying $\|X_0\|_{L^p(\Omega, H)} < \infty$ for some $p \geq 2$. Then SWE (2.15) has a unique mild solution given by

$$X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s)) \, ds + \int_0^t E(t-s)B \, dW(s) \quad a.s. \tag{2.18}$$

for each $t \in [0, T]$. Additionally, if $\|X_0\|_{L^p(\Omega, H^{1/2})} < \infty$ then there exists a constant $K_1(p, \beta, T) \in [0, \infty)$ depending on $p, \beta, T$ such that for any $0 \leq \beta < \frac{1}{2}$

$$\sup_{t \in [0, T]} \|X(t)\|_{L^p(\Omega, H^\beta)} \leq K_1(\|X_0\|_{L^p(\Omega, H^\beta)} + 1). \tag{2.19}$$
Replacing $E(t)$ by (2.17) one can write (2.18) as

\[
\begin{aligned}
  u(t) &= C(t)u_0 + \Lambda^{-\frac{1}{2}} S(t)v_0 + \int_0^t \Lambda^{-\frac{1}{2}} S(t-s)F(u(s))ds + O_t, \\
  v(t) &= -\Lambda^{\frac{1}{2}} S(t)u_0 + C(t)v_0 + \int_0^t C(t-s)F(u(s))ds + \hat{O}_t,
\end{aligned}
\]

(2.20)

where $t \in [0, T]$ and we used the notations

\[
O_t = \int_0^t \Lambda^{-\frac{1}{2}} S(t-s) dW(s), \quad \hat{O}_t = \int_0^t C(t-s) dW(s).
\]

(2.21)
Consider the spatial discretizations of (2.15) by a spectral Galerkin method. To this end, for \( N \in \mathbb{N} \) we define a finite dimensional subspace of \( U \) by \( U_N := \text{span}\{e_1, e_2, \cdots, e_N\} \), and the projection operator \( P_N: \dot{H}^\alpha \rightarrow U_N \) by

\[
P_N \xi = \sum_{i=1}^{N} \langle \xi, e_i \rangle e_i, \quad \forall \xi \in \dot{H}^\alpha, \; \alpha \geq -1. \tag{3.1}
\]

The definition of \( P_N \) immediately implies

\[
\| P_N \varphi \|^2 = \left\| \sum_{i=1}^{N} \langle \varphi, e_i \rangle e_i \right\|^2 = \sum_{i=1}^{N} |\langle \varphi, e_i \rangle|^2 \leq \sum_{i=1}^{\infty} |\langle \varphi, e_i \rangle|^2 = \| \varphi \|^2, \quad \forall \varphi \in U. \tag{3.2}
\]
We emphasize that $U_N$ here is chosen as the linear space spanned by the $N$ first eigenvectors of $\Lambda$. This ensures easy simulations of the stochastic convolutions in the proposed numerical schemes (see below). Now we define $\Lambda_N: U_N \rightarrow U_N$ by

$$\Lambda_N \xi = \Lambda P_N \xi = P_N \Lambda \xi = \sum_{i=1}^{N} \lambda_i \langle \xi, e_i \rangle e_i, \quad \forall \xi \in U_N.$$  \hfill (3.3)

Similarly, one can define $\Lambda_N^\gamma : U_N \rightarrow U_N$, $\gamma \in \mathbb{R}$ in $U_N$ as

$$\Lambda_N^\gamma \xi := \sum_{i=1}^{N} \lambda_i^\gamma \langle \xi, e_i \rangle e_i, \quad \xi \in U_N.$$
Applying the spectral Galerkin method to (2.15) gives finite-dimensional stochastic differential equations in $H_N := U_N \times U_N$

\[
\begin{aligned}
\left\{ \begin{array}{l}
dX^N(t) = A_N X^N(t)dt + F_N(X^N)dt + B_N dW(t), \quad t \in (0, T], \\
X^N(0) = X_0^N,
\end{array} \right.
\end{aligned}
\] (3.4)

where $X_0^N = (P_Nu_0, P_Nv_0)^T$ and

\[
X^N = \begin{bmatrix} u^N \\ v^N \end{bmatrix}, \quad A_N = \begin{bmatrix} 0 & I \\ -\Lambda_N & 0 \end{bmatrix},
\]

\[
F_N(X^N) = \begin{bmatrix} 0 \\ P_N F(u^N) \end{bmatrix}, \quad B_N = \begin{bmatrix} 0 \\ P_N \end{bmatrix}.
\]
Analogously, the operator $A_N$ is the generator of a strongly continuous semigroup $E_N(t)$, $t \geq 0$ on $U_N \times U_N$ and

$$E_N(t) = e^{tA_N} = \begin{bmatrix} C_N(t) & \Lambda_N^{-\frac{1}{2}}S_N(t) \\ -\Lambda_N^{\frac{1}{2}}S_N(t) & C_N(t) \end{bmatrix},$$

where $C_N(t) = \cos(t\Lambda_N^{\frac{1}{2}})$ and $S_N(t) = \sin(t\Lambda_N^{\frac{1}{2}})$ for $t \geq 0$ are the cosine and sine operators defined in $U_N$. 
It can be verified straightforwardly that

\[ C_N(t)P_N\varphi = C(t)P_N\varphi = P_N C(t)\varphi, \]

\[ S_N(t)P_N\varphi = S(t)P_N\varphi = P_N S(t)\varphi \]

for \( \varphi \in \dot{H}^\alpha, \alpha \geq -1 \). The following result ensures a unique global solution of (3.4).
Theorem 2

Assume that all conditions in Theorem 1 are fulfilled. Then (3.4) has a unique solution given by

\[ X^N(t) = E_N(t)X_0^N + \int_0^t E_N(t - s)F_N(X^N(s))ds \]
\[ + \int_0^t E_N(t - s)B_NdW(s) \quad a.s. \]  

for any \( t \in [0, T] \). Additionally, there exists a constant \( K_2(\beta, p, T) \) depending on \( \beta, p, T \) such that for any \( 0 \leq \beta < \frac{1}{2} \) and \( t \in [0, T] \),

\[ \| X^N(t) \|_{L^p(\Omega, H^\beta)} \leq K_2(\| X_0 \|_{L^p(\Omega, H^\beta)} + 1). \]  

(3.8)
Similarly to (2.20), (3.7) can be rewritten as

\[
\begin{align*}
\begin{cases}
    u^N(t) &= C_N(t)u_0^N + \Lambda_N^{-\frac{1}{2}} S_N(t)v_0^N \\
    &+ \int_0^t \Lambda_N^{-\frac{1}{2}} S_N(t-s)P_N F(u^N(s))\,ds + \mathcal{O}_t^N,
    \\
    v^N(t) &= -\Lambda_N^{\frac{1}{2}} S_N(t)u_0^N + C_N(t)v_0^N \\
    &+ \int_0^t C_N(t-s)P_N F(u^N(s))\,ds + \hat{\mathcal{O}}_t^N,
\end{cases}
\end{align*}
\]

(3.9)

where for simplicity of presentation we denote \( u_0^N = P_N u_0 \), \( v_0^N = P_N v_0 \) and

\[
\begin{align*}
\mathcal{O}_t^N &= \int_0^t \Lambda_N^{-\frac{1}{2}} S_N(t-s)P_N \,dW(s), \\
\hat{\mathcal{O}}_t^N &= \int_0^t C_N(t-s)P_N \,dW(s).
\end{align*}
\]

(3.10)
The following lemma is an immediate consequence of (2.9), (2.19) and (3.8).

**Lemma 3**

Assume that all conditions in Theorem 1 are fulfilled, and let \( u(t) \) and \( u^N(t) \) be given by (2.20) and (3.9), respectively. Then there exists a constant \( K_3(p,T,L,\beta) \) depending on \( p,T,L,\beta \) such that

\[
\| F(u(t)) \|_{L^p(\Omega,U)} + \| F(u^N(t)) \|_{L^p(\Omega,U)} \leq K_3(\| X_0 \|_{L^p(\Omega,H)} + 1), \quad t \in [0,T].
\]

(3.11)
Armed with the above preparations, we are now able to analyze the spatial discretization error.

**Theorem 4 (Spatial discretization error)**

Suppose that all conditions in Theorem 1 are satisfied. Then it holds for all \( t \in [0, T] \) that

\[
\| u^N(t) - u(t) \|_{L^2(\Omega, U)} \leq K_4 \left( \| X_0 \|_{L^2(\Omega, H^{1/2-\epsilon})} + 1 \right) N^{-\frac{1}{2}+\epsilon} \tag{3.12}
\]

for arbitrarily small \( \epsilon > 0 \), where \( u(t) \) and \( u^N(t) \) are given by (2.20) and (3.9), respectively, and where \( K_4(\epsilon, T) \in [0, \infty) \) is a constant depending on \( \epsilon, T \).
Based on the spatial approximation (3.9), we propose two time-stepping schemes as follows:

\[
\begin{align*}
\left\{
\begin{array}{l}
u^{N}_{m+1} & = -\Lambda_{N}^{1/2} S_{N}(\tau) u^{N}_{m} + C_{N}(\tau) v^{N}_{m} + \Lambda_{N}^{-1} (I - C_{N}(\tau)) P_{N} F(u^{N}_{m}) \\
& \quad + \int_{t_{m}}^{t_{m+1}} \Lambda_{N}^{-1/2} S_{N}(t_{m+1} - s) P_{N} dW(s), \\
\end{array}
\right.
\]

\[
\begin{align*}
u^{N}_{m+1} & = -\Lambda_{N}^{1/2} S_{N}(\tau) u^{N}_{m} + C_{N}(\tau) v^{N}_{m} + \Lambda_{N}^{-1} (I - C_{N}(\tau)) P_{N} F(u^{N}_{m}) \\
& \quad + \int_{t_{m}}^{t_{m+1}} C_{N}(t_{m+1} - s) P_{N} dW(s),
\end{align*}
\]

(3.13)
and

\[
\begin{align*}
    u^N_{m+1} &= C_N(\tau) u^N_m + \Lambda_N^{-\frac{1}{2}} S_N(\tau) v^N_m + \tau \Lambda_N^{-\frac{1}{2}} S_N(\tau) P_N F(u^N_m) \\
    &\quad + \int_{t_m}^{t_{m+1}} \Lambda_N^{-\frac{1}{2}} S_N(t_{m+1} - s) P_N \, dW(s), \\
    v^N_{m+1} &= -\Lambda_N^{\frac{1}{2}} S_N(\tau) u^N_m + C_N(\tau) v^N_m + \tau C_N(\tau) P_N F(u^N_m) \\
    &\quad + \int_{t_m}^{t_{m+1}} C_N(t_{m+1} - s) P_N \, dW(s),
\end{align*}
\]  

(3.14)

for \( m = 0, 1, 2, \cdots, M - 1 \). Here \( u^N_m \) and \( v^N_m \) are, respectively, the temporal approximations of \( u^N(t) \) and \( v^N(t) \) at the grid points \( t_m = m\tau \), with the initial values \( u^N_0 = P_N u_0, \ v^N_0 = P_N v_0 \), and \( \tau = T/M \) being the stepsize.
Implementation of the schemes

It is worthwhile to point out that both proposed schemes are much easier to simulate than it appears at first sight. To show this fact, we take the scheme (3.13) for example and make some remarks on its implementation. Observe first that, for $i = 1, 2, \cdots, N$, $m = 0, 1, \cdots, M - 1$,

\[
\zeta^i_m := \left\langle \int_{t_m}^{t_{m+1}} \Lambda_N^{-1} S_N(t_{m+1} - s) P_N \, dW(s), e_i \right\rangle \\
= \lambda_i^{-\frac{1}{2}} \int_{t_m}^{t_{m+1}} \sin\left((t_{m+1} - s)\lambda_i^{\frac{1}{2}}\right) \, d\beta_i(s)
\]
are mutually independent normally distributed random variables satisfying

\[ \mathbb{E}[\zeta_m^i] = 0, \quad \text{Var}(\zeta_m^i) = \mathbb{E}[|\zeta_m^i|^2] = \frac{1}{2\lambda_i} \left( \tau - \frac{\sin(2\tau\sqrt{\lambda_i})}{2\sqrt{\lambda_i}} \right). \]

Similarly, for \( i = 1, 2, \cdots, N, \ m = 0, 1, \cdots, M - 1, \)

\[ \hat{\zeta}_m^i := \left\langle \int_{t_m}^{t_{m+1}} C_N(t_{m+1} - s) P_N \, dW(s), e_i \right\rangle \]

\[ = \int_{t_m}^{t_{m+1}} \cos \left( (t_{m+1} - s) \lambda_i^{\frac{1}{2}} \right) \, d\beta_i(s) \]

are mutually independent normally distributed random variables with
\[ \mathbb{E}[\hat{\zeta}_i^m] = 0, \quad \text{Var}(\hat{\zeta}_m) = \mathbb{E}[|\hat{\zeta}_m|^2] = \frac{1}{2} \left( \tau + \frac{\sin(2\tau \sqrt{\lambda_i})}{2\sqrt{\lambda_i}} \right). \]

Moreover, the covariance of \( \zeta_m \) and \( \hat{\zeta}_m \) are given by

\[ \text{Cov}(\zeta_m, \hat{\zeta}_m) = \mathbb{E}[\zeta_m \hat{\zeta}_m] = \frac{1 - \cos(2\tau \sqrt{\lambda_i})}{4\lambda_i} \]

for \( i = 1, 2, \cdots, N, \) \( m = 0, 1, \cdots, M - 1. \)
Let $D^i_m$ be a family of $2 \times 2$ matrices with

$$Q^i_m = \begin{bmatrix}
\text{Var}(\zeta^i_m) & \text{Cov}(\zeta^i_m, \hat{\zeta}^i_m) \\
\text{Cov}(\zeta^i_m, \hat{\zeta}^i_m) & \text{Var}(\hat{\zeta}^i_m)
\end{bmatrix} = D^i_m(D^i_m)^T. \quad (3.15)$$

Then the pair of correlated normally distributed random variables $(\zeta^i_m, \hat{\zeta}^i_m)^T$ can be determined by two independent standard normally distributed random variables:
\[
\begin{bmatrix}
\hat{\zeta}_i \\
\hat{\hat{\zeta}}_i
\end{bmatrix}
= D_m^i
\begin{bmatrix}
R_i^m \\
\hat{R}_i^m
\end{bmatrix},
\] (3.16)

where \( R_m^i : \Omega \to \mathbb{R}, \hat{R}_m^i : \Omega \to \mathbb{R} \) for \( i = 1, 2, \cdots, N \) and \( m = 0, 1, \cdots, M \) are independent, standard normally distributed random variables. Accordingly, the components of \( u_m^N \) and \( v_m^N \) in (3.13), i.e., \( \langle u_m^N, e_i \rangle \) and \( \langle v_m^N, e_i \rangle \) for \( i = 1, 2, \cdots, N \) and \( m = 0, 1, \cdots, M \), can be calculated by the following recurrence equations:
\[
\langle u_{m+1}^N, e_i \rangle = \cos(\tau \sqrt{\lambda_i}) \langle u_m^N, e_i \rangle + \lambda_i^{-\frac{1}{2}} \sin(\tau \sqrt{\lambda_i}) \langle v_m^N, e_i \rangle + \lambda_i^{-1}(1 - \cos(\tau \lambda_i^{\frac{2}{i}})) \langle F(u_m^N), e_i \rangle + \zeta^i_m, \quad (3.17)
\]
\[
\langle v_{m+1}^N, e_i \rangle = -\lambda_i^{\frac{1}{2}} \sin(\tau \sqrt{\lambda_i}) \langle u_m^N, e_i \rangle + \cos(\tau \sqrt{\lambda_i}) \langle v_m^N, e_i \rangle + \lambda_i^{-\frac{1}{2}} \sin(\tau \lambda_i^{\frac{2}{i}}) \langle F(u_m^N), e_i \rangle + \hat{\zeta}^i_m. \quad (3.18)
\]
\[
u_{m+1}^N = \langle u_{m+1}^N, e_1 \rangle e_1 + \langle u_{m+1}^N, e_2 \rangle e_2 + \cdots + \langle u_{m+1}^N, e_N \rangle e_N,
\]
\[
u_{m+1}^N = \langle v_{m+1}^N, e_1 \rangle e_1 + \langle v_{m+1}^N, e_2 \rangle e_2 + \cdots + \langle v_{m+1}^N, e_N \rangle e_N.
\]
Theorem 5

Suppose that the nonlinear function \( f \) in (1.1) satisfies (1.2) and (1.3), and let \( W(t) \) be the cylindrical I-Wiener process represented by (2.6). Moreover, assume that \( \| u_0 \|_{L^p(\Omega, \dot{H}^1)} + \| v_0 \|_{L^p(\Omega, \dot{H}^0)} < \infty \) for all \( p \in [2, 4] \). Let \( u(t) \) be the mild solution of (1.1) represented by (2.20) and let \( u^N_m \) be the numerical approximation produced by (3.13) or (3.14), with \( \tau = \frac{T}{M} \) being the time stepsize. Then it holds for all \( m = 0, 1, 2, \cdots, M \) and for arbitrarily small \( \epsilon > 0 \) that

\[
\| u^N_m - u(t_m) \|_{L^2(\Omega, U)} \leq K \left( N^{-\frac{1}{2} + \epsilon} + \tau^{1-\epsilon} \right),
\]

where \( K \in [0, \infty) \) is a constant depending on \( T, \epsilon, L \) and the initial data.
The mean-square approximation error \((4.1)\) is composed of two parts. The first term is due to the spatial discretization and the second term is caused by the temporal discretization. The detailed proof of Theorem 5 depends on the following lemmas.

**Lemma 6**

*Assume that \(S(t)\) and \(C(t)\) are the sine and cosine operators as defined above. Then for all \(\gamma \in [0, 1]\) there exists some constant \(\hat{c}(\gamma)\) such that*

\[
\| (S(t) - S(s)) \Lambda^{-\frac{\gamma}{2}} \|_{\mathcal{L}(U)} \leq \hat{c}(t - s)^\gamma,
\]

\[
\| (C(t) - C(s)) \Lambda^{-\frac{\gamma}{2}} \|_{\mathcal{L}(U)} \leq \hat{c}(t - s)^\gamma
\]

*for all \(t \geq s \geq 0\).*
Next, a regularity result on the stochastic process $u^N(t)$ is derived, which plays an important role in obtaining the strong convergence rate of the proposed schemes.

**Lemma 7**

Assume that all conditions in Theorem 5 are fulfilled. Then for $\beta \in [0, \frac{1}{2})$ and $\delta \in [0, \frac{1}{2}]$ there exists a constant $K_5(\beta, \delta, p, T) > 0$, depending on $\beta, \delta, p, T$ and the initial data, such that

$$
\|u^N(t) - u^N(s)\|_{L^p(\Omega, \dot{H}^{-\delta})} \leq K_5 (t - s)^{\beta + \delta}, \quad t > s, \ t, s \in [0, T].
$$

(4.3)
Lemma 8

Let $F : U \to U$ be the Nemytskij operator defined by (2.4), with the conditions (1.2) and (1.3) fulfilled. Then for $\alpha \in (0, \frac{1}{2})$ and $\delta \in (\frac{1}{2}, 1]$ there exists a constant $K_6(\alpha, \delta)$ such that

$$\|F'(\varphi)\psi\|_\alpha \leq K_6(\|\varphi\|_\alpha + 1)\|\psi\|_\delta$$

(4.4)

holds for all $\varphi \in \dot{H}^\alpha$, $\psi \in \dot{H}^\delta$. 
Consider the Sine-Gordon equation driven by space-time white noise:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} - \sin(u) + \dot{W}, \quad t \in (0, 1], \ x \in (0, 1), \\
u(0, x) &= \frac{\partial u}{\partial t}(0, x) = 0, \ x \in (0, 1), \\
u(t, 0) &= u(t, 1) = 0, \ t > 0.
\end{align*}
\]

(5.1)

The corresponding deterministic equation is used to describe the dynamics of coupled Josephson junctions driven by a fluctuating current source. Next, we use various numerical schemes to solve (5.1) and compare their computational errors. Note that the expectations are approximated by computing averages over 100 samples.
Figure 1: Spatial errors for the spectral Galerkin method applied to SWE (5.1).
Figure 2: Strong convergence rates in time for various time integrators applied to (3.4).
Figure 3: Strong convergence rates in time for various time integrators applied to (3.4).
Example 2

Consider nonlinear SWE

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{1+u}{1+u^2} + \dot{W}, \quad 0 < t \leq 1, \quad x \in (0, 1), \\
u(0, x) &= 0, \quad \frac{\partial u}{\partial t}(0, x) = 1, \quad x \in (0, 1), \\
u(t, 0) &= u(t, 1) = 0, \quad t > 0.
\end{aligned}
\] (5.2)

Subsequently, we focus on the overall computational efforts of various fully discrete schemes, with the spectral Galerkin discretizations in space. We take the number of realizations of independent random variables needed for approximations as a measure for the computational effort.
Recall that the Crank-Nicolson-Maruyama (CNM) scheme and the stochastic trigonometric method (STM) converge with rate $\frac{1}{3} - \epsilon$ and order $\frac{1}{2} - \epsilon$ in time, respectively. In space, the spectral Galerkin method converges with order $\frac{1}{2} - \epsilon$ for arbitrarily small $\epsilon > 0$. In order to balance the errors in space and in time, we set $M = N^{\frac{3}{2}}$ for CNM scheme and $M = N$ for STM. Similarly, we set $M = N^{\frac{1}{2}}$ for the schemes (3.13) and (3.14).
With these settings, the four schemes all result in an overall approximation error $O(N^{-\frac{1}{2}}+\epsilon)$. The overall approximation errors produced by different schemes are listed in Table 1-3.

**Table 1**: Computational errors of the Crank-Nicolson-Maruyama scheme with $M = N^{3/2}$

<table>
<thead>
<tr>
<th>$N = 2^2$</th>
<th>$N = 2^4$</th>
<th>$N = 2^6$</th>
<th>$N = 2^8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.13058</td>
<td>0.065411</td>
<td>0.032987</td>
<td>0.016622</td>
</tr>
</tbody>
</table>
Table 2: Computational errors of the stochastic trigonometric method with $M = N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>0.054405</th>
<th>0.037954</th>
<th>0.026312</th>
<th>0.017867</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^7$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^9$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Computational errors of the schemes (3.13) and (3.14), with $M = N^{1/2}$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$2^4$</th>
<th>$2^6$</th>
<th>$2^8$</th>
<th>$2^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scheme (3.13)</td>
<td>0.055098</td>
<td>0.027929</td>
<td>0.01372</td>
<td>0.0068861</td>
</tr>
<tr>
<td>Scheme (3.14)</td>
<td>0.05617</td>
<td>0.028007</td>
<td>0.013708</td>
<td>0.0068762</td>
</tr>
</tbody>
</table>
Given a precision $\varepsilon = 0.02$, we are to compare the required computational costs for the above four schemes.

- **Crank-Nicolson-Maruyama scheme:**
  \[
  2^{20} = 1048576(N = 2^8, M = 2^{12}) \text{ random variables.}
  \]

- **Stochastic trigonometric method:**
  \[
  2^{18} = 262144(N = M = 2^9) \text{ random variables.}
  \]

- (3.13) and (3.14): $2 \times 2^{12} = 8192(N = 2^8, M = 2^4)$ random variables.
As discussed above, with the same precision, the proposed schemes (3.13) and (3.14) can reduce the number of used random variables greatly and improve the computational efficiency significantly.
Figure 4: The overall approximation errors against number of used random variables.
**Figure 5:** The overall approximation errors against number of used random variables.
Figure 6: The overall approximation errors against number of used random variables.
Figure 7: The overall approximation errors against number of used random variables.
Semilinear stochastic wave equation (SWE) driven by additive space-time white noise

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x, u) + \dot{W}, \quad t \in (0, T], \ x \in (0, 1),
\end{aligned}
\]

\[
\begin{aligned}
  u(0, x) = u_0(x), \ & \frac{\partial u}{\partial t}(0, x) = v_0(x), \ x \in (0, 1), \\
  u(t, 0) = u(t, 1) = 0, \ & t > 0,
\end{aligned}
\]

\[f: [0, 1] \times \mathbb{R} \to \mathbb{R}\] is a smooth nonlinear function satisfying

\[
|f(x, z)| \leq L(|z| + 1),
\]

\[
|\frac{\partial f}{\partial z}(x, z)| \leq L, \quad |\frac{\partial^2 f}{\partial x \partial z}(x, z)| \leq L, \quad \text{and} \quad |\frac{\partial^2 f}{\partial z^2}(x, z)| \leq L
\]
Schemes for SWEs

Spatial discretization: spectral Galerkin method;

Temporal approximation: exponential time integrators.

Convergence results

\[ \| u_N^m - u(t_m) \|_{L^2(\Omega,U)} \leq K \left( N^{-\frac{1}{2} + \epsilon} + \tau^{1-\epsilon} \right), \]
Future work

- Convergence under non-global Lipschitz coefficients;
- Weak convergence;
- Convergence under multiplicative noise;
- Stiffness arising from spatial semi-discretization of SWEs;
- How numerical schemes preserve the properties of SWEs?
- How the methods are implemented rapidly?


Thanks!